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# Determinantal Criterion of Hopf Bifurcations

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**Abstract**—In this note, we present a criterion of a class of Hopf bifurcations using the properties of bialternate products of matrices.

## 1. Introduction

In theoretical economic dynamics, it is highly desirable to have a criterion to check the stability of a system without solving its characteristic equation (e.g., Gandolfo [4], Chap. 16).

Given a linear dynamic system of differential equations, the system is asymptotically stable if and only if all the roots of the characteristic equation have negative real parts. As such a stability criterion, the criterion of Routh-Hurwitz is well known (e.g., Gantmacher [5]). Given a parametric system of differential equations, the system has a simple Hopf bifurcation when a pair of complex conjugate roots of the characteristic equation passes through the imaginary axis while all other roots have negative real roots. Liu [8] showed the criterion of a simple Hopf bifurcation based on the criterion of Routh-Hurwitz.

The criterion of Routh-Hurwitz requires the sign of the Hurwitz determinants whose elements are the coefficients of the characteristic polynomial. Fuller [3] gave the alternative stability criterion involving the determinants whose elements are simpler than the Hurwitz determinants. In this note, the criterion of a simple Hopf bifurcation based on the criterion of Fuller will be shown.

## 2. The Routh-Hurwitz and Fuller criteria

Consider a system

$$\frac{dz_i}{dt} = \sum_{j=1}^{n} a_{ij} z_i, \quad (i = 1, 2, \ldots, n), \tag{1}$$

where $a_{ij}$ are real constant coefficients. In vector-matrix notation, the system is

$$\frac{dz}{dt} = Az, \quad z \in \mathbb{R}^n. \tag{2}$$

For the system to be asymptotically stable it is necessary and sufficient that all the eigenvalues of the characteristic equation of matrix $A$,

$$|\lambda I_n - A| = 0, \tag{3}$$

have negative real parts (e.g., Gantmacher [5]).

Let us denote the characteristic equation (3) as a polynomial equation

$$p_n\lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_0 = 0, \quad p_n > 0. \tag{4}$$

Then the Routh-Hurwitz criterion can be stated as follows.

**Theorem 1 (e.g., Gantmacher [5]).** *All the roots of* (4) *have negative real parts if and only if*

$$H_1 > 0, H_2 > 0, \ldots, H_n > 0, \tag{5}$$

*where $H_i$ are the Hurwitz determinants:*

$$\begin{aligned}
H_1 &= p_{n-1},\\
H_2 &= \begin{vmatrix} p_{n-1} & p_{n-3} \\ p_n & p_{n-2} \end{vmatrix},\\
H_3 &= \begin{vmatrix} p_{n-1} & p_{n-3} & p_{n-5} \\ p_n & p_{n-2} & p_{n-4} \\ 0 & p_{n-1} & p_{n-3} \end{vmatrix},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
H_n &= \begin{vmatrix} p_{n-1} & p_{n-3} & p_{n-5} & \cdots & 0 \\ p_n & p_{n-2} & p_{n-4} & \cdots & 0 \\ 0 & p_{n-1} & p_{n-3} & \cdots & 0 \\ 0 & p_n & p_{n-2} & \cdots & 0 \\ & \cdots & & \cdots & \\ & \cdots & & \cdots & p_0 \end{vmatrix}.
\end{aligned} \tag{6}$$

Next, according to Fuller [3], an alternative stability criterion will be stated.

**Theorem 2 (Routh [11]).** *Consider* (4)*. Let*

$$q_m\mu^m + q_{m-1}\mu^{m-1} + \cdots + q_0 = 0 \tag{7}$$

*be the equation of root-pair-sums of* (4)*, i.e., let the roots of* (7) *be $n(n-1)/2$ $(= m)$ values*

$$\mu = \lambda_i + \lambda_j, \quad (i = 2, 3, \ldots, n;\ j = 1, 2, \ldots, i-1), \tag{8}$$

*where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the roots of* (4)*. Let $q_m > 0$. Then for the roots of* (4) *to have all their real parts negative it is necessary and sufficient that $p_0, p_1, \ldots, p_{n-1} > 0$ and $q_0, q_1, \ldots, q_{m-1} > 0$.*

Further, a matrix whose characteristic equation is the equation (7) is given in terms of the bialternate product studied by Stéphanos [12] (cited by Fuller).

**Definition 1 (Stéphanos [12]).** Let $A$ be an $n$-dimensional matrix $(a_{ij})$ and $B$ be an $n$-dimensional matrix $(b_{ij})$. Let $F$ be an $m = n(n-1)/2$-dimensional matrix $(f_{pq,rs})$ whose rows are labelled $pq$ $(p = 2, 3, \ldots, n; q = 1, 2, \ldots, p-1)$, whose columns are labelled $rs$ $(r = 2, 3, \ldots, n; s = 1, 2, \ldots, r-1)$ and whose elements are

$$f_{pq,rs} = \frac{1}{2}\left[\begin{vmatrix} a_{pr} & a_{ps} \\ b_{qr} & b_{qs} \end{vmatrix} + \begin{vmatrix} b_{pr} & b_{ps} \\ a_{qr} & a_{qs} \end{vmatrix}\right]. \tag{9}$$

Then $F$ is the bialternate product of $A$ and $B$, and is written as $A \cdot B$.

**Theorem 3 (Stéphanos [12]).** *The characteristic roots of the matrix*

$$G = 2A \cdot I_n, \tag{10}$$

*where $A$ is an $n$-dimensional matrix $(a_{ij})$ and $I_n$ is an $n$-dimensional identity matrix, are the $n(n-1)/2$ values*

$$\lambda_i + \lambda_j, \quad (i = 2, 3, \ldots, n;\ j = 1, 2, \ldots, i-1), \tag{11}$$

*where $\lambda_i$ are eigenvalues of matrix $A$.*

The element of $G$ is

$$g_{pq,rs} = \begin{vmatrix} a_{pr} & a_{ps} \\ \delta_{qr} & \delta_{qs} \end{vmatrix} + \begin{vmatrix} \delta_{pr} & \delta_{ps} \\ a_{qr} & a_{qs} \end{vmatrix} \tag{12}$$

where $\delta_{ij}$ is the Kronecker delta, $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$. With $p > q$ and $r > s$ (12) is

$$g_{pq,rs} = \begin{cases} -a_{ps} & \text{if} \quad r = q \\ a_{pr} & \text{if} \quad r \neq p \ \text{ and } \ s = q \\ a_{pp} + a_{qq} & \text{if} \quad r = p \ \text{ and } \ s = q \\ a_{qs} & \text{if} \quad r = p \ \text{ and } \ s \neq q \\ -a_{qr} & \text{if} \quad s = p \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

**Theorem 4 (Fuller [3]).** *Let $A = (a_{ij})$ be a real square matrix of dimension $n > 1$. Let $G = (g_{pq,rs})$ be the square matrix of dimension $m = n(n-1)/2$ defined by* (10)*, and elements given by* (12) *or equivalently by* (13)*. Then for the characteristic roots of $A$ to have all their real parts negative, it is necessary and sufficient that in the characteristic polynomial of $A$,*

$$|\lambda I_n - A|, \tag{14}$$

*and in the characteristic polynomial of $G$,*

$$|\mu I_m - G|, \tag{15}$$

*the coefficients of $\lambda^i$ $(i = 0, 1, \ldots, n-1)$ and $\mu^i$ $(i = 0, 1, \ldots, m-1)$ should all be positive.*

The elements of the determinants in the Fuller criterion are simpler functions of the $a_{ij}$ than those in the Routh-Hurwitz criterion. In fact, the Hurwitz determinants have elements which are themselves sums of determinants.

Let us note that, although the criterion of Routh-Hurwitz is well known in theoretical economic dynamics, there exists a few application of the criterion of Fuller to dynamic economic systems. As for applications of the latter, see Murata [10].

Arapostathis and Jury [1] showed that the criteria in Theorem 2 can be reduce to $1 + n(n-1)/2$ criteria.

**Theorem 5 (Arapostathis and Jury [1]).** *Consider* (4)*,* (7)*, and* (8)*. Then for the roots of* (4) *to have their real parts negative, it is necessary and sufficient that $p_0 > 0$ and $q_0, q_1, \ldots, q_{m-1} > 0$.*

## 3. Hopf Bifurcations and the Liu criterion

Following Guckenheimer and Holmes [6] and Liu [8], simple Hopf bifurcations and the Liu criterion are stated. Consider a system

$$\frac{dz}{dt} = f_\alpha(z),\ z \in \mathbb{R}^n,\ \alpha \in \mathbb{R}^1, \tag{16}$$

with an equilibrium $(z^*, \alpha^*)$, and $f \in C^\infty$. Assume that

(i) The Jacobian matrix $D_z f_{\alpha^*}(z^*)$ has a simple pair of purely imaginary eigenvalues and other eigenvalues have negative real parts.

Then there is a smooth curve of equilibria $(z(\alpha), \alpha)$ with $z(\alpha^*) = z^*$. The eigenvalues $\lambda(\alpha), \bar{\lambda}(\alpha)$ of $J(\alpha) = D_z f_\alpha(z(\alpha))$ which are purely imaginary at $\alpha = \alpha^*$ vary smoothly with $\alpha$. Moreover, If

(ii) $\dfrac{d}{d\alpha}(\mathrm{Re}\ \lambda(\alpha^*)) \neq 0,$

then there is a *simple* Hopf bifurcation. The *simple* Hopf bifurcations is used to distinguish from the Hopf bifurcations with some other eigenvalues with non-zero real parts.

Let us denote the characteristic equation of the Jacobian matrix $J(\alpha)$,

$$|\lambda I_n - J(\alpha)| = 0, \tag{17}$$

as a polynomial equation

$$p(\lambda; \alpha) = p_n(\alpha)\lambda^n + p_{n-1}(\alpha)\lambda^{n-1} + \cdots + p_0(\alpha) = 0,\ p_n(\alpha) > 0, \tag{18}$$

where every $p_i(\alpha)$ is a smooth function of $\alpha$.

**Theorem 6 (Liu [8]).** *Assume there is a smooth curve of equilibria $(z(\alpha), \alpha)$ with $z(\alpha^*) = z^*$ for the system* (16)*. Conditions (i) and (ii) for a simple Hopf bifurcation are equivalent to the following conditions on the coefficients of the characteristic polynomial $p(\lambda; \alpha)$:*

*(i')* $p_0(\alpha^*) > 0, H_1(\alpha^*) > 0, H_2(\alpha^*) > 0, \ldots, H_{n-2}(\alpha^*) > 0, H_{n-1}(\alpha^*) = 0,$

*(ii')* $\dfrac{d}{d\alpha}(H_{n-1}(\alpha^*)) \neq 0,$

*where $H_i$ are the Hurwitz determinants.*

The Liénard-Chipard criterion is well known as another stability criterion (e.g., Gantmacher [5]). As Gantmacher said, the Liénard-Chipard criterion has advantage over the Routh-Hurwitz criterion in the sense that the number of determinantal inequalities in the former is roughly half of that in the latter. From the practical point of view, Manfredi and Fanti [9] reformulated the Liu criterion by replacing the criteria based on the Routh-Hurwitz criterion with the corresponding Liénard-Chipard criteria.

It is worthwhile to point out that the coefficient criterion of Hopf bifurcations with some other eigenvalues with non-zero real parts has been established for the first few values of $n$. Asada and Yoshida [2] showed such a criterion for $n = 4$. To our knowledge, one for $n \geqslant 5$ is not available in the literature.

## 4. Alternative Criterion

An alternative criterion of simple Hopf bifurcations based on the criterion of Fuller will be given.

The following theorem is an immediate consequence of Theorem 2 and the idea of the proof is due to Routh [11]'s proof of Theorem 2 and Liu [8]'s proof of Theorem 6.

**Theorem 7.** *Consider* (4), (7), *and* (8). *Let* $q_m > 0$. *Then, in order that* (4) *has a simple pair of purely imaginary roots and other roots have negative real parts, it is necessary and sufficient that* $p_0, p_1, \ldots, p_{n-1} > 0, (n-1 \neq 1)$ *and* $q_0 = 0,\ q_1, q_2, \ldots, q_{m-1} > 0$.

*Proof.* $\Rightarrow$: If (4) has a simple pair of purely imaginary roots and other roots have negative real parts, the left hand side of (4) can be written as

$$p(\lambda) = r(\lambda)(\lambda^2 + s_1\lambda + s_0), \tag{19}$$

where $s_0 > 0, s_1^2 - 4s_0 < 0, s_1 = 0$,

$$r(\lambda) = r_{n-2}\lambda^{n-2} + r_{n-3}\lambda^{n-3} + \cdots + r_0, \quad r_{n-2} > 0, \tag{20}$$

all $(n-2)$ roots of $r(\lambda)$ have negative real parts. Thus, from Theorem 2, the coefficients of $r(\lambda)$, $r_0, r_1, \ldots, r_{n-2}$ are all positive. Hence all the coefficients of $p(\lambda)$ are positive, i.e., $p_0, p_1, \ldots, p_{n-1} > 0$, except for $p_1 = 0$ for $n = 2$. On the other hand, from the relation (8), the left hand side of (7) can be expressed as

$$q(\mu) = t(\mu)(\mu + u_0), \tag{21}$$

where $u_0 = 0$,

$$t(\mu) = t_{m-1}\mu^{m-1} + t_{m-2}\mu^{m-2} + \cdots + t_0, \quad t_{m-1} > 0, \tag{22}$$

all $(m-1)$ roots of $t(\mu)$ have negative real parts. By the same reasoning above, the coefficients of $t(\mu)$, $t_0, t_1, \ldots, t_{m-1}$ are all positive. Hence we have $q_0 = 0$, $q_1, q_2, \ldots, q_{m-1} > 0$. For $n = 2$, $p_1 = s_1 = u_0 = q_0$ by definition.

$\Leftarrow$: If $p_0, p_1, \ldots, p_{n-1} > 0, (n-1 \neq 1)$ hold, $p(\lambda) > 0$ for $\lambda \geqslant 0$, i.e., (4) has no positive or zero real roots. On the other hand, if $q_0 = 0,\ q_1, q_2, \ldots, q_{m-1} > 0$ hold, from the factorisation (21), (7) has all its real roots in the open left half plane except for one zero real root. But from (8), the real roots of (7) include twice the real parts of the complex roots of (4). Hence (4) has a simple pair of purely imaginary roots and other roots have negative real parts. □

Suppose the roots of the characteristic equation (17) are $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then there is the equation

$$q(\mu;\alpha) = q_m(\alpha)\mu^m + q_{m-1}(\alpha)\mu^{m-1} + \cdots + q_0(\alpha) = 0, q_m(\alpha) > 0, \tag{23}$$

whose roots are given by the $n(n-1)/2$ values

$$\mu = \lambda_i + \lambda_j, \quad (i = 2, 3, \ldots, n;\ j = 1, 2, \ldots, i-1), \tag{24}$$

whose degree is

$$m = 1 + 2 + \cdots + (n-1) = \frac{1}{2}n(n-1), \tag{25}$$

and every $q_i(\alpha)$ is a smooth function of $\alpha$. Let us define

$$G(\alpha) = 2J(\alpha) \cdot I_n, \tag{26}$$

where $J(\alpha)$ is the Jacobian matrix of the system (16), $I_n$ is an $n$-dimensional identity matrix, and $\cdot$ denotes bialternate product. Then $G(\alpha)$ is the matrix whose characteristic equation is the equation (23).

**Theorem 8.** *Assume there is a smooth curve of equilibria* $(z(\alpha), \alpha)$ *with* $z(\alpha^*) = z^*$ *for the system* (16). *Conditions (i) and (ii) for a simple Hopf bifurcation are equivalent to the following conditions in the characteristic polynomial* $p(\lambda;\alpha)$ *and the associated polynomial* $q(\mu;\alpha)$*:*

*(i")* $p_0(\alpha^*), p_1(\alpha^*), \ldots, p_{n-1}(\alpha^*) > 0, (n-1 \neq 1),\ q_0(\alpha^*) = 0,\ q_1(\alpha^*), q_2(\alpha^*), \ldots, q_{m-1}(\alpha^*) > 0$,

*(ii")* $\dfrac{d}{d\alpha}(q_0(\alpha^*)) \neq 0$.

*Proof.* (i) $\Leftrightarrow$ (i"): This follows by applying Theorem 7 to (18) and (23) at $\alpha = \alpha^*$. In this case, we have the relations (19)–(22) and denote these as (19')–(22') for future reference.

(ii) $\Leftrightarrow$ (ii"): Let the eigenvalues of $J(\alpha)$ which are purely imaginary at $\alpha = \alpha^*$ be $\lambda^0(\alpha)$ and $\overline{\lambda}^0(\alpha)$. From (21') and (24), Re $\lambda^0(\alpha) = (\lambda^0(\alpha) + \overline{\lambda}^0(\alpha))/2 = \mu^0(\alpha)/2 = -u_0(\alpha)/2$. This relation, $t_0(\alpha) > 0$, and $u_0(\alpha^*) = 0$ give

$$\frac{d}{d\alpha}(\mathrm{Re}\ \lambda^0(\alpha^*)) \gtreqless 0 \Leftrightarrow -\frac{1}{2}\left[\frac{d}{d\alpha}(u_0(\alpha^*))\right] \gtreqless 0 \Leftrightarrow$$
$$-\frac{1}{2}\left[\frac{d}{d\alpha}(t_0(\alpha^*)u_0(\alpha^*))\right] \gtreqless 0 \Leftrightarrow -\frac{1}{2}\left[\frac{d}{d\alpha}(q_0(\alpha^*))\right] \gtreqless 0. \tag{27}$$

□

This theorem is an immediate consequence of Theorems 1, 4, and 6. In addition, Fuller mentioned the necessary condition for the characteristic equation (14) to have at least one pair of purely imaginary roots in investigating the boundary of the stability region, which is expressed as $\det G = 0$. In Guckenheimer et al. [7], this condition is used for the numerical detection of candidates of Hopf bifurcation points.

It must be noted that the condition (i”) involves redundant criteria as well as Theorem 4. The following theorem is an immediate consequence of Theorem 5 and the idea of the proof is due to Araposthathis and Jury [1]’s proof of Theorem 5.

**Theorem 9.** *Consider* (4), (7), *and* (8). *Let* $q_m > 0$. *Then, in order that* (4) *has a simple pair of purely imaginary roots and other roots have negative real parts, it is necessary and sufficient that* $p_0 > 0$ *and* $q_0 = 0,\ q_1, q_2, \ldots, q_{m-1} > 0$.

*Proof.* $\Rightarrow$: Necessity follows from Theorem 7.

$\Leftarrow$: If $p_0 > 0$, (4) has no zero real roots, and we deduce that (4) can either have an even number of positive real roots or no positive real roots at all. But if $q_0 = 0,\ q_1, q_2, \ldots, q_{m-1} > 0$, (7) has all its real roots in the open left half plane except for one zero real root. Thus, from (8), the real roots of (4) must be negative. Also, the real roots of (7) include twice the real parts of the complex roots of (4). Hence (4) has a simple pair of purely imaginary roots and other roots have negative real parts. □

**Theorem 10.** *The condition (i”) in Theorem 8 can be reduced to the following:*

*(i”’)* $p_0(\alpha^*) > 0,\ q_0(\alpha^*) = 0, q_1(\alpha^*), q_2(\alpha^*), \ldots, q_{m-1}(\alpha^*) > 0.$

*Proof.* (i) $\Leftrightarrow$ (i”’): This follows directly from Theorem 9. □

## 5. Example

**Example 1.** For the system (16) with $n = 3$, the Jacobian matrix is

$$J(\alpha) = \begin{bmatrix} a_{11}(\alpha) & a_{12}(\alpha) & a_{13}(\alpha) \\ a_{21}(\alpha) & a_{22}(\alpha) & a_{23}(\alpha) \\ a_{31}(\alpha) & a_{32}(\alpha) & a_{33}(\alpha) \end{bmatrix}, \tag{28}$$

and the matrix defined by (22) is

$$G(\alpha) = \begin{bmatrix} a_{22}(\alpha) + a_{11}(\alpha) & a_{23}(\alpha) & -a_{13}(\alpha) \\ a_{32}(\alpha) & a_{33}(\alpha) + a_{11}(\alpha) & a_{12}(\alpha) \\ -a_{31}(\alpha) & a_{21}(\alpha) & a_{33}(\alpha) + a_{22}(\alpha) \end{bmatrix}. \tag{29}$$

The non-redundant criteria for a simple Hopf bifurcation are

$$\text{(i”)} \quad -\det J(\alpha^*) > 0,\ -\operatorname{tr} J(\alpha^*) > 0,\ -\det G(\alpha^*) = 0, \tag{30}$$

$$\text{(ii”)} \quad \frac{d}{d\alpha}(-\det G(\alpha^*)) \neq 0. \tag{31}$$

The above non-redundant criteria are obtained by the similar procedure in Fuller [3].

## 6. Conclusion

The determinantal criterion of simple Hopf bifurcations is provided in Theorem 8.

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