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Global Dynamics in a Heterogeneous Duopoly Model with Noninvertible Maps

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Abstract—This paper investigates the coexisting attractors and their basin boundaries in nonlinear economic dynamics with noninvertible maps. To this end, it construct a complementary two-good economy model in which each of the goods is produced by a representative firm. This paper examines that chaotic fluctuation from which the both firms can benefit more than from a stationary point has a robustness regarding the initial conditions. Through numerical simulations, it shows that if 3-periodic chaotic attractors coexist, basins of the attracting sets of benefitial chaos are structurally stable and symmetric between the firms.

1. Introduction

Intensive analyses of nonlinear economic dynamics in last two decades show that the output adjustment of oligopolistic firms can be chaotic. Rand [6] demonstrates a possibility of complex dynamics in a simple duopoly model with unimodal reaction functions. Puu [5] also Kopel [3] give microeconomic foundations of chaotic dynamics of the duopoly model in different ways. Matsumoto and Nonaka [4] investigate statistical properties of chaotic Cournot model with complementary goods and demonstrates that the firms can benefit from chaotic output adjustment more than from a stationary point.

This paper extend the analysis of Matsumoto and Nonaka by dealing with attractors and their basins of chaotic output adjustment. Then it examines that the chaotic trajectories along which the firms can get higher profits more than staying at a stationary point have robustness regarding their initial conditions. If chaotic trajectories converge to 3-periodic attractors, the basins of attracting set of the beneficial chaos are structurally stable and symmetric between the firms.

2. Model

Consider a two-good economy in which the goods are complement. The two goods, namely x and y are respectively produced by representative firms, say firm 1 and firm 2. Under the consideration of complementarity the firms have *positive external effects* on their demand, then the inverse demand functions of the goods are given by

$$\begin{aligned} p_1(x,y) &= (\alpha-1)^2 - \frac{1}{2}x + (\alpha y)^2 \\ p_2(x,y) &= 1 - \frac{1}{2}y + (\beta x)^2 \end{aligned} \tag{1}$$

here p_1 and p_2 are the market price of good $x \in X$ and $y \in Y$ and X and Y are strategy spaces of firm 1 and firm 2.

In this paper we also assume that firms interact on their production behavior, say, the firms have *negative external effects* on their supply. Here we introduce simple linear correlation between the goods, then the cost functions of the firms are given by

$$\begin{aligned} C_1(x,y) &= 2\alpha(\alpha-1)yx \\ C_2(x,y) &= 2\beta xy \end{aligned} \tag{2}$$

here C_i is the cost function of firm i. (2) implies that the marginal cost of each firm is constant to its own output but linearly increasing to the other's. We assume that $\alpha \geq 1$ and $\beta \geq 0$ for nonnegative value of the marginal costs.

Profits of the two firms are defined by subtracting the costs from their revenues. The profit function of firm i, Π_i is respectively given by

$$\begin{aligned} \Pi_1(x,y) &= p_1 x - C_1(x,y), \\ \Pi_2(x,y) &= p_2 y - C_2(x,y). \end{aligned} \tag{3}$$

At each discrete time period t the two firms produce the outputs namely x_t and y_t. The firm adjust their outputs observing the other's behavior at the end of every period. Assuming that the firms decide their production at period $t+1$ in order to maximize their expected profits, then the output adjustment takes place as following dynamics,

$$\begin{aligned} x_t &= r_1(y_{t+1}^{(e)}) \equiv \arg\max \Pi_1(x, y_{t+1}^{(e)}), \\ y_t &= r_2(x_{t+1}^{(e)}) \equiv \arg\max \Pi_2(x_{t+1}^{(e)}, y) \end{aligned} \tag{4}$$

were $y_{t+1}^{(e)}$ represents the expectation of firm 1 about the output of firm 2 and $x_{t+1}^{(e)}$ does so the expectation of firm 2 about firm 1's output. The map $r_1 : Y \to X$ and $r_2 : X \to Y$ are generally called *reaction functions* and determined as

$$\begin{aligned} r_1(y) &= (\alpha y - \alpha + 1)^2, \\ r_2(x) &= (\beta x - 1)^2. \end{aligned} \tag{5}$$

We apply *bounded rational* (or *naive*) expectation method on each firm, then assume $y^{(e)}_{t+1} = y_t$ and $x^{(e)}_{t+1} = y_x$. Therefore suppose that the two-dimensional map $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$T(x, y) \equiv (r_1(y), r_2(x)), \tag{6}$$

then the time evolution of the complementary goods economy is obtained by $T(x, y) = (x', y')$ where $'$ represents the one-period advancement operator.

Thus given an initial condition (i.c.) $(x_0, y_0) \in X \times Y$, a trajectory of the economy $\{x_t, y_t\}$ is given by the iteration of the map T,

$$\{x_t, y_t\} = \{T^t(x_0, y_0)\} \tag{7}$$

where $T^t = T \circ T^{t-1}$ and T^0 is an identical map.

To investigate the long-run behavior of the output adjustment in (4) we will check the general properties of the two-dimensional map T. To recall generic dynamics of T we use following one-dimensional maps, F and G such that

$$\begin{aligned} F(x) &\equiv r_1 \circ r_2(y), \\ G(y) &\equiv r_2 \circ r_1(x). \end{aligned} \tag{8}$$

3. Basic Results

Here we briefly summarize the result of Matsumoto and Nonaka [4]. Denote the graph of reaction curves $x = r_1(y)$ and $y = r_2(x)$ by R_1 and R_2,

$$\begin{aligned} R_1 &\equiv \{(r_1(y), y) | y \in Y\}, \\ R_2 &\equiv \{(x, r_2(x)) | x \in X\}. \end{aligned} \tag{9}$$

As far as $\alpha \le 2$ and $\beta \le 2$, both r_1 and r_2 map the unit interval $I \equiv [0, 1]$ to itself and then $T : I \times I \to I \times I$. If the i.c. (x_0, y_0) belongs $R_1 \cup R_2$, then the two firms moves alternately, i.e. (x_t, y_t) belongs alternately R_1 and R_2. This means that the graph of the union of the two reaction functions is a trapping set for T, i.e. $T(R_1 \cup R_2) \subset R_1 \cup R_2$.[1] The cycles of the two-dimensional map T are related to those of the one-dimensional maps F and G. The analysis of Bischi et al. [1] say that if F (also G) has a stable cycle of period n, then T has a stable cycle of period $2n$ its periodic points alternately belong to R_1 and R_2. Thus as far as we restrict the i.c. $(x_0, y_0) \in R_1 \cup R_2$, the dynamic properties of T can be characterized by F or G. Figure 1 illustrates the bifurcation diagram of F with respect to $1 \le \alpha \le 2$ and $0 \le \beta \le 2$. In Fig.1, red-coloured area depicts stable region and as the colour is getting change, the number of period of a stable cycle increases doublingly. In white-coloured area, we can observe chaotic fluctuations of the output adjustment. We can see that the behavior of the output adjustment changes from convergence to a stable stationary point to chaotic fluctuation.

[1]In fact we can check that if i.c. $(x_0, y_0) \in R_1 \cup R_2$, then $T^t(x_0, y_0) \in R_1 \cup R_2$ $\forall t \ge 0$.

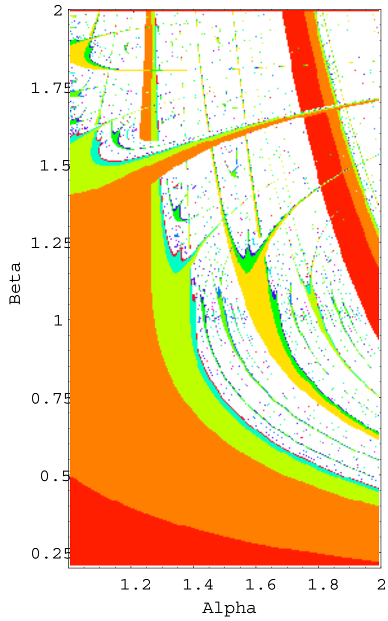

Figure 1: Bifurcation diagram of F

To investigate the long-run behavior of the output adjustment and its implication to the economy, we calculate the average profit of the two firms taken along the trajectory of the output adjustment. first we restrict the i.c. $(x_0, y_0) \in R_1 \cup R_2$ and get following results.

- For symmetric case ($\alpha = \beta$), one firm can benefit from chaotic fluctuation of the output adjustment more than from the stationary point but the other does not.

- If α is relatively close to 1 and β is relatively close to 2, both of the firms can benefit from chaotic fluctuation more than a stationary point.

4. Critical Curve Analysis

Now, we investigate generic case which includes the i.c. $(x_0, y_0) \notin R_1 \cup R_2$. Here T may have coexisting attractors and the output adjustment faces multistability. Therefore we examine first the structure of the coexisting attractors and basins of the attracting sets.

The point $(x', y') = T(x, y)$ is called the *rank-1 image* of (x, y). A point (x, y) such that $T(x, y) = (x', y')$ is called a *rank-1 preimage* of (x', y'). If distinct two point $(x, y) \neq (\hat{x}, \hat{y})$ have the same image, $T(x, y) = T(\hat{x}, \hat{y}) = (x', y')$, then the map T is said to be noninvertible. For $\alpha > 1$ and $\beta > 0$, r_1 and r_2 then T becomes noninvertible maps. To examine the attractors and their basins of a two-dimensional map, we introduce the analysis of *criti-*

cal curves LC.[2] LC is the image of the fold curve LC_{-1}. LC_{-1} coincides with the set of points in which the Jacobian determinant of T vanishes. Suppose that J is the Jacobian matrix of T,

$$J = \begin{pmatrix} 0 & r_1'(y) \\ r_2'(x) & 0 \end{pmatrix}, \tag{10}$$

then a point of LC_{-1} satisfies $\det J = 0$. Therefore we have $LC_{-1} = LC_{-1}^{(x)} \cup LC_{-1}^{(y)}$ where

$$\begin{aligned} LC_{-1}^{(x)} &= \{(x,y)|x = \tfrac{1}{\beta},\ y \in X\}, \\ LC_{-1}^{(y)} &= \{(x,y)|x \in X,\ y = \tfrac{1-\alpha}{\alpha}\}. \end{aligned} \tag{11}$$

LC and its rank-k images determine the structure of the attracting sets of T. If T has chaotic attractors then the iteration of $T^t(LC_{-1})$ determines the boundaries of the attracting sets.

Let A be an attracting set of T. The basins of A, $\mathcal{B}(A)$ consists of non-connecting rectangles in phase space $I \times I$. The basin boundaries of all coexisting attractors are determined by periodic points of unstable cycles and their rank-k preimages. Suppose that C_u is an unstable of T. In the case of the two-dimensional map T given by (6), the basin boundaries are horizontal and vertical lines crossing all points of $T^{-k}(C_u)$.

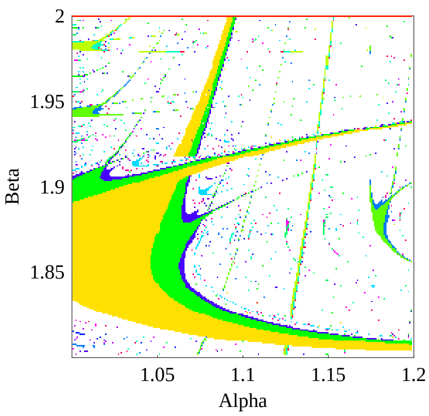

Figure 2: Bifurcation windows with 3-period cycles

5. 3-periodic Chaos

Here we determine the parameter region in which chaotic trajectories starting from $(x_0, y_0) \notin R_1 \cup R_2$ are beneficial for the both firms. In this investigation we focus of the area in which the chaotic trajectories starting from $(x_0, y_0) \in R_1 \cup R_2$ are beneficial. We can see that in this area there exists a bifurcation window of period 3 which is illustrated in Figure 2. In yellow-coloured regions of Fig. 2, F has cycles of period 3 and in green-coloured regions it has 6-periodic cycles. We can see from Fig. 2 the cycles of period 3 become chaotic attractors through a period-doubling bifurcation process.

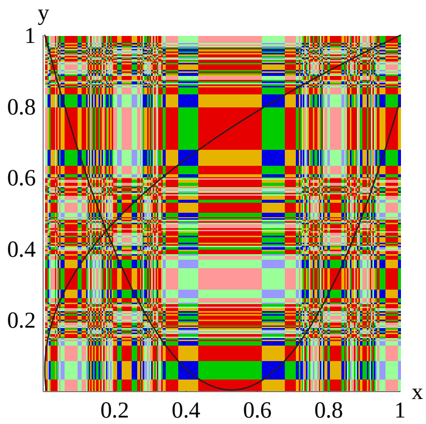

Figure 3: Basins of attraction of seven cycles

If $\alpha = 1.05$ and $\beta = 1.905$ F (also G) has two stable cycles of period 3 and T has seven coexisting stable cycles.[3] Figure 3 illustrates the basins of attractions of seven stable cycles. In Fig. 3, the rectangles of same colour are non-connecting basins of same cycles. Here we can see that the basins of attracting periodic points have *box within a box* structure.

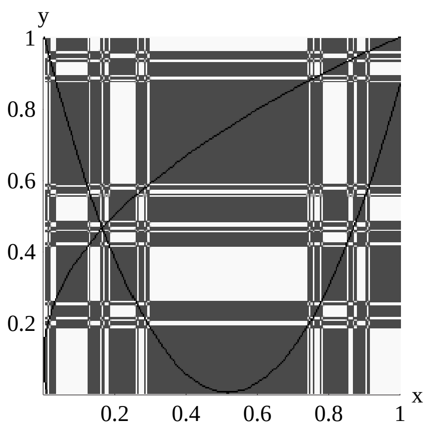

Figure 4: Basins of attraction of beneficial chaos

With this parameter combination, the T has a unique stationary point in $I \times I$. For both firms we calculate the stationary profit and the average profits with the seven cycles then examine which cycles generates higher profit more than the stationary point. Figure 4 shows the basins of " beneficial cycles". The cycles to which the trajectories starting from black-coloured regions converge generate

[2] The notion of "critical curve" is the two-dimensional version of n-dimensional critical manifold and LC comes from the French "Ligne Critique."

[3] Here T has two cycles of period 3 and four cycles of period 6.

higher profit more than the stationary point and the structure of the basins are symmetric between the two firms.

If we increase the both of the parameter values then periodic trajectories becomes chaotic. Just after becoming chaotic, the output adjustment still has "periodic structure." That is, the chaotic trajectories of the output adjustment are 3-periodic and the attractor consist of three non-connecting subsets of the phase space.

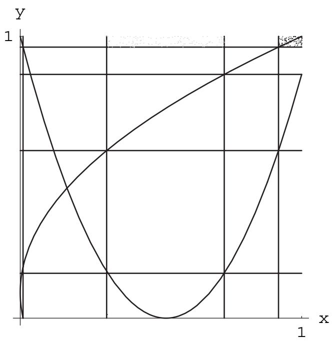

Figure 5: 3-periodic chaotic attractors

Let $\alpha = 1.1$ and $\beta = 1.93$. Then two 3-periodic chaotic attractors coexist. Figure 5 illustrates the chaotic attractors with 3-periodic structures. The borders of the attractors are determined by the rank-k image of critical curve LC i.e. $\{T^k(LC_{-1})\}$ $k \geq 1$. For both two firms the trajectories converging to the black-coloured periodic attractor generate higher profits more than the stationary profits. In addition, the basins of the beneficial chaos have same structure with the case of stable cycles of period-3 and are symmetric between the two firms. The output adjustment keeps these properties as far as the chaotic attractors are periodic. Thus they are structurally stable, too.

If the parameter values increase further the chaotic attractors of the output adjustment loss their 3-periodic structure. To determine the parameter region in which the output adjustment is chaotic and the attractors are 3-periodic we calculate the periodic point of unstable 3-period cycle of T and compare them with the points of $LC = T(LC_{-1})$. Figure 6 illustrates the parameter region of 3-periodic chaos. In the white-coloured rhombic area enclosed by two bold lines and *periodic arms*, the output adjustment converges to 3-periodic chaotic attractors and then holds the above properties.

6. Discussion

The first numerical results summarised in Section 3 shows that if we focus the i.c. on the reaction curves (i.e. $(x_0, y_0) \in R_1 \cup R_2$), chaotic output adjustment can be beneficial for both two firms if α is relatively close to 1 and β is relatively close to 2. The next two sections examine the case of $(x_0, y_0) \notin R_1 \cup R_2$). If cycles of F becomes chaotic, then T has infinite chaotic attractors. However if

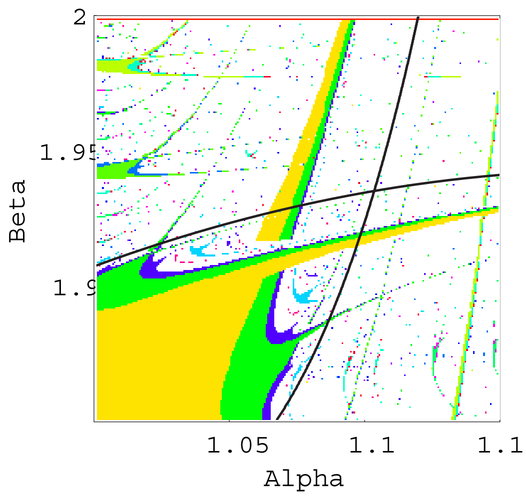

Figure 6: The border of 3-periodic chaos

these chaotic attractors are periodic, their long-run properties are eventually same among the trajectories converging to same attracting set. Numerical results demonstrated through Section 5 indicate that with 3-periodic chaotic attractors, the basins of attracting sets which generate higher average profits more than stationary profit are symmetric between the two firms and structurally stable. In this sense, we can say that beneficial chaos of output adjustment has robustness regarding its initial conditions.

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