2004 International Symposium on Nonlinear
Theory and its Applications (NOLTA2004)
Fukuoka, Japan, Nov. 29 - Dec. 3, 2004

Money Supply and Economic Fluctuations

Hiroyuki Yoshida†

†College of Economics, Nihon University
1-3-2 Misaki, Chiyoda-ku, Tokyo 101-8360 Japan
Email: yoshida@eco.nihon-u.ac.jp

Abstract—This paper examines a simple monetary optimizing model with sticky prices. By using the Hopf bifurcation theorem, we show the existence of limit cycles, which mean that the indeterminacy of equilibrium arises. This result suggests that a constant money growth rule might fail to stabilize the economy.

1. Introduction

The analysis of whether the supply of money has a real effect is a central topic in the field of monetary economics. If real economic activity is independent of the rate of money growth, money is said to be "superneutral." The seminal work by [12] shows that the superneutrality of money is valid in the long-run steady state using the money-in-the-utility-function (MIUF) approach. After his contribution, the validity of the Sidrauski result has been reexamined by subsequent studies (for example, [3], [5], [7], [11], and [13] among others).

Most of them have demonstrated that the superneutrality result crucially depends on some particular assumptions of the Sidrauski paper. If the labor supply is endogenous[1] or if Harrod- neutral technological progress is present[2], then the long-run steady state is not in general invariant to the growth rate of money.

From a different angle, the present paper attempts to reexamine the superneutality of money: we allow for nominal price stickiness to extend a Sidrauski-type model. As pointed out by [4, p. 127], prominent twentieth-century economists such as John Maynard Keynes, Milton Friedman, Franco Modigliani, and James Tobin believe that price stickiness plays an important role in explaining economic fluctuations.

The purpose of the present paper is to explore the relation between the monetary growth rate and the qualitative properties of the steady state. In particular, we investigate the possibility of endogenous and persistent business cycles through a sticky-price channel[3]. By choosing the monetary growth rate as a bifurcation parameter, we apply the Hopf bifurcation theorem to our sticky price model. Under certain conditions, we can find that a high money supply growth would generate persistent and periodic fluctuations around the steady state. On the other hand, a low money supply growth would yield a stable stationary point. Thus the equilibrium path is indeterminate.

The remainder of the paper is organized as follows. Section 2 presents a monetary optimizing model with sticky prices and characterizes the dynamic properties of the steady state. Section 3 discusses two examples. Finally, Section 4 concludes the major findings of our analysis.

[1] See, for example, [5, pp. 773–775].

[2] On this subject, see [13].

[3] The emergence of chaotic fluctuations can be found in [10], who assumes a flexible-price environment and constructs a MIUF model in the discrete time context. His result is derived through the Li-York theorem.

2. A monetary economy

In this section we develop a simple infinite-horizon monetary economy with sticky-prices. There exist three agents: a representative household, a representative producer, and the government, and three commodities: physical goods, bonds, and money.

2.1. The model

The infinitely lived representative household maximizes her lifetime utility:

$$\int_0^\infty [\ln c_t + v(m_t)]\, e^{-\rho t} dt, \tag{1}$$

where c_t denotes the level of consumption, m_t represents real money balances, and $\rho\,(> 0)$ is the subjective discount rate. The function, $v(m)$, is well-behaved with $v'(\cdot) > 0$ and $v''(\cdot) < 0$.

In the economy there are only two assets: money (m) and government bonds (b). The household accumulates real financial wealth, $a_t(= m_t + b_t)$, according to the following law of motion:

$$\dot{a}_t = r_t a_t + y_t - \tau_t - c_t - (r_t + \pi_t) m_t, \tag{2}$$

where, r_t is the real interest rate, y_t is real output, $\tau_t(> 0)$ is lump-sum taxes levied by the government (if $\tau_t < 0$, it denotes lump-sum transfer payments from the government), and π_t is the rate of inflation.

The household's optimization problem is to choose c_t and m_t to maximize (1) subject to (2). The optimality solution is obtained by setting up the Hamiltonian function:

$$H = \ln c + v(m) + \mu[ra + y - \tau - c - (r + \pi)m], \tag{3}$$

where μ is the costate variable. The optimality conditions are:

$$\partial H/\partial c = 0 : 1/c = \mu, \tag{4}$$

$$\partial H/\partial m = 0 : v\prime(m) = \mu(r + \pi), \tag{5}$$

$$\dot{\mu} = \rho\mu - \partial H/\partial a : \dot{\mu} = \mu(\rho - r). \tag{6}$$

Consolidating the optimality conditions, we can obtain the following Euler equation,

$$\dot{c}_t = [v'(m_t)c_t - \pi_t - \rho]c_t. \tag{7}$$

The governmental budget constraint is

$$\dot{a}_t = r_t a_t + g - \tau_t - (r_t + \pi_t)m_t, \tag{8}$$

where g denotes exogenous and constant government consumption. We assume that the government controls the level of lump-sum taxes to peg the level of nominal government liabilities; $A_t = p_t a_t = A_0$, for all $t \geq 0$, where p_t is the price level[4]. Hence, the government budget constraint (8) can be rewritten as

$$\tau_t - g = (r_t + \pi_t)b_t. \tag{8'}$$

Equation (8') implies that the primary surplus, $\tau_t - g$, is equal to interest payments on the public debt, $(r_t + \pi_t)b_t$, at all times. Furthermore, we assume that the government maintains a constant growth rate of nominal money supply:

$$\dot{M} = \theta M, \tag{9}$$

where θ is a constant parameter. This policy rule, therefore, implies that

$$\dot{m} = (\theta - \pi)m. \tag{10}$$

As for the firm's production decision, the output level is determined by the effective demand:

$$y_t = c_t + g. \tag{11}$$

This equation captures a Keynesian feature: the flexible quantity adjustment leads the goods market into equilibrium at each point in time. Furthermore, we introduces price stickiness. The inflation rate adjusts to the gap between effective demand and capacity output. Specifically,

$$\dot{\pi}_t = \beta(c_t + g - y_n),\ \beta > 0, \tag{12}$$

where y_n denotes capacity output, whose level is constant and exogenous. On this point, see [6].

2.2. The analysis

We now consider the local stability properties of the long-run steady state. Equations (7), (12), and (10) constitute a complete system of nonlinear dynamic equations with three endogenous variables (c, m, π). The long-run steady state is defined as a set of constant functions $\{c, m, \pi\}$ satisfying (7), (12), and (10); that is,

$$v'(m_*)c_* = \theta + \rho, \tag{13}$$

$$c_* + g = y_n, \tag{14}$$

$$\pi_* = \theta. \tag{15}$$

¿From the above equations, we can see the superneutrality of money in the long-run steady state; the steady state level of consumption is independent of money growth.[5] We should furthermore note that the acceleration of money supply raises the inflation rate and thereby lowers the amount of real money balances demanded, namely

$$m_* = m_*(\theta),\ dm_*(\theta)/d\theta = 1/(v''(m_*)c_*) < 0. \tag{16}$$

Linearizing (7), (12), and (10) around the steady state leads to the dynamic system,

$$\begin{pmatrix} \dot{c} \\ \dot{m} \\ \dot{\pi} \end{pmatrix} = \begin{bmatrix} v'c_* & v''c_*^2 & -c_* \\ 0 & 0 & -m \\ \beta & 0 & 0 \end{bmatrix} \begin{pmatrix} c - c_* \\ m - m_* \\ \pi - \theta \end{pmatrix}. \tag{17}$$

The corresponding characteristic equation of this system, $P(\lambda) = \lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0$, has the following coefficients:[6]

$$b_1 = -\text{trace}J = -v'(m_*)c_* < 0, \tag{18}$$

$$\begin{aligned} b_2 &= \text{sum of all second-order principal minors of } J \\ &= \beta c_* > 0, \end{aligned} \tag{19}$$

$$b_3 = -\det J = \beta v''(m_*)m_* {c_*}^2 < 0, \tag{20}$$

$$\Delta := b_1 b_2 - b_3 = v'(m_*)(\eta(m_*) - 1)\beta {c_*}^2, \tag{21}$$

where J is the Jacobian matrix on the right-hand side of (17) and $\eta(m_*)$ represents the coefficient of relative risk aversion evaluated at the steady state m_*;

$$\eta(m_*) = -\frac{m_* v''(m_*)}{v'(m_*)} > 0. \tag{22}$$

By utilizing $\eta(m_*)$, we can characterize the equilibrium dynamics of our model:

Proposition 1 *If $\eta(m_*) < 1$, then the equilibrium path is the steady state itself. On the other hand, if $\eta(m_*) > 1$, then there exists a continuum of perfect-foresight equilibria.*

[4] This assumption is adopted by [6].

[5] Naturally we have to impose $g < y_n$ to ensure the positive amount of c_*.

[6] The coefficients of the characteristic equation can be computed through the elements of the Jacobian matrix. On this point see, for example, [2, p.633] and [8, pp. 247–248].

Proof. Since $b_3 < 0$, the characteristic equation, $P(\lambda) = 0$, has at least one positive root, say, $\lambda_1 > 0$. Then, we have

$$\lambda_2\lambda_3 > 0, \qquad (23)$$

on the ground that $b_3 = -\lambda_1\lambda_2\lambda_3 < 0$.

Using the following relations:[7]

$$b_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 > 0 \qquad (24)$$

$$b_1b_2 - b_3 = -(\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_1), \qquad (25)$$

we can derive

$$\Delta(= b_1b_2 - b_3) = -[(\lambda_1)^2 + b_2](\lambda_2 + \lambda_3). \qquad (26)$$

Equation (26) implies that $b_1b_2 - b_3$ and $\lambda_2 + \lambda_3$ have the opposite sign, since $(\lambda_1)^2 + b_2 > 0$. Thus, recalling (23), we can show that both real parts of λ_2 and λ_3 are positive (negative) if $\Delta(= b_1b_2 - b_3)$ is negative (positive).[8]

These results lead to the conclusion that the characteristic equation, $P(\lambda) = 0$, has three positive roots for all $\eta(m_*) \in (0, 1)$, while it has two negative roots and one positive root for $\eta(m_*) > 1$. This completes the proof of the proposition, since we have two control variables, c and m, in our model. ■

If $\eta(m_*) < 1$, the equilibrium path is determinate. By contrast, if $\eta(m_*) > 1$, the equilibrium displays indeterminacy, which can be connected with sunspots or business cycles. Proposition 1 therefore suggests that proper and adequate control of θ stabilizes the economy so long as the government maintains $\eta(m_*(\theta)) < 1$. We proceed with our analysis under the following hypothesis:

Hypothesis 1 (Increasing relative risk aversion). The relative risk aversion $\eta(\cdot)$ is an increasing function of m: $d\eta(m)/dm > 0$.

This hypothesis is explained and justified by Arrow in his argument. [See [1, pp. 96–98].] Under Hypothesis 1, we can obtain the following corollary:

Corollary 1 *Suppose that there exists a critical value, $\theta = \theta_H$, such that $\eta(m_*(\theta_H)) = 1$. If $\theta > \theta_H$, then the equilibrium path is the steady state itself. On the other hand, if $\theta < \theta_H$, then there exists a continuum of perfect-foresight equilibria.*

However, this assertion is not always the case, because our analysis is limited to the neighborhood of the steady state. Let us now examine further this point, which is our main concern. Concentrating on the parameter θ, we can prove the following proposition.

Proposition 2 *Suppose that there exists a critical value, $\theta = \theta_H$, such that $\eta(m_*(\theta_H)) = 1$. Then the dynamic system undergoes a Hopf bifurcation, which generates persistent fluctuations.*

Proof. To apply the Hopf bifurcation theorem, we have to prove that (i) the characteristic equation has a pair of pure imaginary roots and no other roots with zero real parts, and (ii) the real part of the pure imaginary roots is not stationary with respect to θ. For our purpose, it suffices to verify that[9] $b_1(\theta_H) \neq 0$, $b_2(\theta_H) > 0$, $\Delta(\theta_H) = b_1b_2 - b_3 = 0$, and $d\Delta(\theta_H)/d\theta \neq 0$.

The first three conditions are obvious from (18), (19) and (21). Differentiating (21) at θ_H yields

$$\left.\frac{d\Delta(\theta)}{d\theta}\right|_{\theta=\theta_H} = v'(m_*)\beta {c_*}^2\theta \left.\frac{d\eta(m_*)}{dm_*}\frac{dm_*(\theta)}{d\theta}\right|_{\theta=\theta_H}. \qquad (27)$$

Thus, we can confirm that $d\Delta(\theta_H)/d\theta < 0$ since $d\eta(m_*)/dm_* > 0$ and $dm_*(\theta)/d\theta < 0$. This completes the proof. ■

Proposition 2 establishes only the existence of limit cycles. To obtain more meaningful results, we impose two additional assumptions: increasing relative risk aversion and supercritical bifurcation. The latter presupposes that a stable limit cycle appears when the steady state is unstable.[10] Then we can obtain:

Corollary 2 *In Proposition 2, we adopt the hypothesis of increasing relative risk aversion and assume that a supercritical bifurcation occurs. Then there appears a stable limit cycle if $\theta > \theta_H$, whereas the equilibrium path exhibits indeterminacy if $\theta < \theta_H$.*

Proof. When $d\eta(m)/dm > 0$, we have $d\Delta(\theta_H)/d\theta < 0$ from (27). Thus $\Delta(\theta)$ is positive for $\theta < \theta_H$, while it is negative for $\theta > \theta_H$. Accordingly a similar argument as in the proof of Proposition 1 shows that the characteristic equation possesses three positive roots for $\theta > \theta_H$ and two negative roots for $\theta < \theta_H$. Then, by using Proposition 2 and the assumption of supercritical bifurcation, we immediately arrive at the conclusion of the corollary. ■

In this case, two types of indeterminacy can be found according to θ. As discussed in Proposition 1, the equilibrium path is indeterminate if $\eta(m_*) > 1$ $(\theta < \theta_H)$. This is because the steady state is a stable point. The other type of indeterminacy arises for $\eta(m_*) < 1$ $(\theta > \theta_H)$; there exists an infinite number of trajectories diverging from the

[7] On this point, see [2, p.634].

[8] Note that this statement is valid regardless of whether both λ_2 and λ_3 are real or complex numbers.

[9] For details, see [2, pp. 634–635].

[10] In general, there are two types of Hopf bifurcation. In the case of supercritical bifurcation, a stable limit cycle occurs around the unstable equilibrium. On the other hand, the subcritical bifurcation yields an unstable limit cycles, which surrounds the stable equilibrium point. For further details, see, for example, [9].

steady state and thereafter converging to the stable limit cycle. Such cyclical trajectories can be supported as equilibrium paths since they fulfill all the optimality and the market clearance conditions. Accordingly, from the viewpoint of stabilization policy, we reasonably conclude that none of the growth rates of money supply can stabilize the economy.

3. Concluding Remarks

In this paper we construct a sticky-price model and characterize the equilibrium dynamics of our model (Proposition 1). Moreover, we prove the existence of persistent fluctuations around the normal equilibrium employing the Hopf bifurcation theorem (Proposition 2). Under additional assumptions, this proposition tells us that the government cannot stabilize the economy if monetary policy takes the form of a constant money growth rule (Corollary 2).

References

[1] K. J. Arrow, *Essays in the theory of risk-bearing*, North-Holland, Amsterdam, 1971.

[2] T. Asada, & W. Semmler, "Growth and finance: An intertemporal model," *Journal of Macroeconomics*, vol.17, pp. 623–649, 1995.

[3] K. Asako, "The utility function and the superneutrality of money on the transition path," *Econometrica*, vol.51, pp. 1593–1596, 1983.

[4] L. Ball, & N. G. Mankiw, "A sticky-price manifesto," *Carnegie-Rochester Conference Series on Public Policy*, vol.41, pp. 127–151, 1994.

[5] W. A. Brock, "Money and growth: The case of long run perfect foresight," *International Economic Review*, vol.15, pp. 750–777, 1974.

[6] W. H. Buiter, & N. Panigirtzoglou, "Liquidity trap: How to avoid them and how to escape them," NBER Working Paper 7245, 1999.

[7] S. Fischer, "Capital accumulation on the transition path in a monetary optimizing model," *Econometrica*, vol.47, pp. 1433–1439, 1979.

[8] G. Gandolfo, *Economics Dynamics*, Springer, Berlin,1997.

[9] J. Guckenheimer, & P. Holmes, *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, Springer, New York, 1983.

[10] K. Matsuyama, "Endogenous price fluctuations in an optimizing model of a monetary economy," *Econometrica*, vol. 59, pp. 1617–1631, 1991.

[11] M. Obstfeld, "Multiple stable equilibria in an optimizing perfect-foresight model," *Econometrica*, vol.52, pp. 223–228, 1984.

[12] M. Sidrauski, "Rational choice and patterns of growth in a monetary economy, " *American Ecoomic Review*, vol.57, pp. 534-544, 1967.

[13] J. J. Siegel,"Technological change and the superneutrality of money," *Journal of Money, Credit, and Banking*, vol.15, pp. 363–367, 1983