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Extensions of nonautonomous nonlinear integrable systems to higher dimensions

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Abstract—Nonautonomous nonlinear systems are one of exciting subjects in Physical sciences and the Integrable systems. Actually, several nonautonomous partial differential equations from physical sciences are thought of as perturbations of integrable systems. In this manuscript, we discuss nonautonomous higher dimensional integrable systems in detail. Finding new integrable systems is an important but difficult problem in the study of Integrable systems. For the seek of new integrable systems, many researchers have mainly investigated autonomous and lower dimensional nonlinear systems. We present new nonautonomous higher dimensional systems from the Burgers and KdV equations by applying the Painlevé test which determine whether or not a given system is integrable. We also give a Cole-Hopf transformation with variable coefficients for the nonautonomous versions of the Burgers equation in higher dimensions.

1. Introduction

Modern theories of nonlinear science have been highly developed over the last half century. Particularly, the integrable system has attracted great interests of a number of mathematicians and physicists. One of the reason for such attentions is algebraic solvability of the integrable systems. In addition to their theoretical importance, they have remarkable applications that many physical systems are thought of as perturbations of integrable systems such as hydrodynamics, nonlinear optics, plasma and certain field theories and so on [1]. The notion of the integrable systems [2] is frequently not defined precisely but rather is characterized generally by various interrelated common features including the space-localized solutions(solitons), Lax pairs, Bäcklund transformations and some Painlevé properties[3, 4]. Finding new integrable systems is an important, but difficult problem because of their ambiguous definition and undeveloped mathematical background.

For discovery of new integrable systems, many researchers have mainly investigated autonomous and lower dimensional nonlinear systems [5, 6, 7, 8, 9, 10, 11, 12]. Thus many autonomous (1+1)-dimensional integrable systems have been found. On the other hand, there are few research studies to find nonautonomous nonlinear integrable systems, since they are essentially complicated and their theory is still in its early stages. In physical systems, however, nonautonomous nonlinear integrable equations are one of exciting subjects in Integrable systems[13, 14, 15]. Analysis of higher dimensional system is also an active topic in the integrable system. For example, Boiti *et al.* introduced the dromions which are the exponentially localized solutions in two dimensional spaces [16]. Since then, the study of higher dimensions has attracted much more attention. So our goal in this manuscript is to extend nonautonomous integrable equations to higher-dimensions by applying the Painlevé test.

2. Investigation of nonautonomous system via the Painlevé test

In this section, we first give a brief review both of the Painlevé property and of the Painlevé test. Next we present nonautonomous higher dimensional systems of the Burgers and KdV equations by using Painlevé test. It is widely-known that the Painlevé test in the sense of Weiss-Tabor-Carnevale(WTC) method [3, 4] is a powerful tool for investigating autonomous and nonautonomous integrable equations.

2.1. the Painlevé test

Experience has shown that, when a system possesses the Painlevé property, one is integrable. Weiss *et.al.*[3] said that a partial differential equation(PDE) has the Painlevé property when the solutions of the PDE are single-valued about the movable, singularity manifold. They have proposed a technique which determine whether or not a given system is integrable, that we call WTC's method:

When the singularity manifold is determined by

$$\phi(z_1, \cdots, z_n) = 0, \tag{1}$$

and $u = u(z_1, \cdots, z_n)$ is a solution of PDE given, then we assume that

$$u = \phi^\alpha \sum_{j=0}^{\infty} u_j \phi^j, \tag{2}$$

where $\phi = \phi(z_1, \cdots, z_n)$, $u_j = u_j(z_1, \cdots, z_n)$, $u_0 \neq 0$ are analytic functions of z_j in a neighborhood of the manifold (1) and α is a negative integer (so-called the leading order). Substitution of (2) into the PDE determines the value of α and defines the recursion relations for u_j, $j = 0, 1, 2, \cdots$. When the ansatz (2) is correct, the PDE possesses the Painlevé property and is conjectured to be integrable.

2.2. Nonautonomous Burgers type equation

We here discuss a following form of equation:

$$\begin{aligned} &u_t + a(x,z,t)u + b(x,z,t)u_x + c(x,z,t)u_z \\ &+d(x,z,t)uu_z + e(x,z,t)u_x\partial_x^{-1}u_z \\ &+f(x,z,t)u_{xz} + g(x,z,t) = 0, \end{aligned} \tag{3}$$

where $d(x,z,t) + e(x,z,t) \neq 0$, $f(x,z,t) \neq 0$ and subscripts with respect to independent variables denote partial derivatives, for example, $u_x = \dfrac{\partial u}{\partial x}$, $u_{xz} = \dfrac{\partial^2 u}{\partial x \partial z}$ etc, and $\partial_x^{-1} u := \displaystyle\int^x dx' u(x')$. Here $a(x,z,t), b(x,z,t), \cdots, g(x,z,t)$ are functions of two spatial variables x, z and one temporal variable t. If we choose $a(x,z,t) = b(x,z,t) = c(x,z,t) = g(x,z,t) = 0$ and $d(x,z,t) = e(x,z,t) = f(x,z,t) = 1$, equation (3) is reduced to the (2+1)-dimensional Burgers equation:

$$u_t + uu_z + u_x\partial_x^{-1}u_z + u_{xz} = 0, \tag{4}$$

which, by setting $z = x$, reads the (ordinary) Burgers equation which is widely-known to be linearisable or integrable.

Our main goal here is finding new integrable equations. We apply the Painlevé test to equation (3) and determine the coefficients by conditions from the Painlevé test. The Painlevé test for equation (3) requires elimination of the non-local term. Through operations of division and differentiation, equation (3) is transformed to

$$\begin{aligned} &(ea_x - ae_x)uu_x + (ec_x - ce_x)u_xu_z \\ &+(ed_x - de_x)uu_xu_z + (eg_x - ge_x)u_x \\ &+e(d+e)u_x^2u_z + (ae + eb_x - be_x)u_x^2 - e_xu_tu_x \\ &+eu_xu_{xt} + deuu_xu_{xz} + (ce + ef_x - fe_x)u_xu_{xz} \\ &-egu_{xx} - aeuu_{xx} - eu_tu_{xx} - ceu_{xx}u_z \\ &-deuu_{xx}u_z - efu_{xx}u_{xz} + efu_xu_{xxz} = 0, \end{aligned} \tag{5}$$

where a, b, c, d, e, f and g denote $a(x,z,t), b(x,z,t), c(x,z,t), d(x,z,t), e(x,z,t), f(x,z,t)$ and $g(x,z,t)$ respectively. We assume the following singularity manifold expansion with $\phi = \phi(x,z,t)$ for $u = u(x,z,t)$:

$$u = \phi^\alpha \sum_{j=0}^{\infty} u_j \phi^j, \tag{6}$$

where ϕ and the coefficients u_j are analytic functions of the independent variables x, z, t, and $\phi(x,z,t) = 0$ defines the singularity manifold. By a leading order analysis, substituting

$$u = \phi^\alpha u_0, \tag{7}$$

into equation (5), we obtain $\alpha = -1$ and $u_0 \neq 0$. By the substitution of expansion (7) with $\alpha = -1$ into equation (5), the recursion relations for the u_j are presented as follows,

$$\begin{aligned} &(j-1)(j-2)(j+1)e(x,z,t)f(x,z,t)^2\phi_x^4\phi_z u_j \\ &= F(u_{j-1}, \cdots, u_0, \phi_t, \phi_x, \phi_z, \cdots), \end{aligned} \tag{8}$$

where the explicit dependence on t, x, z of the right-hand side comes from that of the coefficients. It is found that the resonance occur at

$$j = -1, 1, 2. \tag{9}$$

Let us note here that the leading order and resonances are the same result as for a (2+1)-dimensional Burgers equation (4). From recurrence relations, we find

$$j = 0: \quad u_0 = \frac{2f}{d+e}\phi_x, \tag{10}$$

$$\begin{aligned} j = 1: \quad &\frac{8e^2f^2z}{\{d+e\}^4} \times \Big[\{f_z(d+e) - f(d_z + e_z)\}\phi_x^5 \\ &- \{f_x(d+e) - f(d_x+e_x)\}\phi_x^4\phi_x \Big] = 0, \end{aligned} \tag{11}$$

$$\begin{aligned} j = 2: \quad &\frac{1}{(d+e)^5}\Big[4ef^3(d-e)(d+e)^2\{\phi_x^2\phi_{xx}\phi_{xz} \\ &+ \phi_x^2\phi_{xxx}\phi_z - \phi_{xx}^2\phi_x\phi_z - \phi_x^3\phi_{xxz}\} + \cdots \\ &+ f(u_1(d_z+e_z) + 3b_x - 2f_{xz}))\}\big]\phi_x^4 \Big] \\ &= 0, \end{aligned} \tag{12}$$

in lower orders. Now we look into cases pass the Painlevé test. We take into account only following cases

1. $e = 0$, $f \neq 0$,
2. $e \neq 0$, $f \neq 0$, $d = d(t)$, $e = e(t)$, $f = f(t)$,
3. $e \neq 0$, $f \neq 0$, $f = (d+e)\exp h(t)$,

where $h(t)$ is a constant of integration with respect to x and z. It is easily checked that Cases 1 is not determined the leading order and resonances. And it is easy to see that Case 2 is a special case of Case 3. Now we discuss following form of equation for Case 3:

$$\begin{aligned}
&u_t + a(x,z,t)u + b(x,z,t)u_x + c(x,z,t)u_z \\
&+d(x,z,t)uu_z + e(x,z,t)u_x\partial_x^{-1}u_z \\
&+\exp\{h(t)\}\{d(x,z,t) + e(x,z,t)\}u_{xz} \\
&+g(x,z,t) = 0.
\end{aligned} \tag{13}$$

Substituting (6) into equation (13), we find

$$\begin{aligned}
j = 0: &\qquad u_0 = 2\exp\{h(t)\}\phi_x, \\
j = 1: &\qquad u_1 : \text{arbitrary},
\end{aligned}$$

from the recurrence relations (10) and (11). For $j = 2$, we have

$$\begin{aligned}
&4\exp\{2h(t)\}(eb_x - be_x - ae - eh'(t))\phi_x^4 \\
&+4\exp\{2h(t)\}(ec_x - ce_x)\phi_x^3\phi_z \\
&+4\exp\{3h(t)\}e(d-e)(\phi_x^2\phi_{xx}\phi_{xz} - \phi_{xx}^2\phi_x\phi_z \\
&-\phi_x^3\phi_{xxz} + \phi_x^2\phi_{xxx}\phi_z) - 4\exp\{2h(t)\}e_x\phi_x^3\phi_t \\
&+4\exp\{3h(t)\}(ed_x - de_x)\phi_x^2\phi_{xx}\phi_z = 0.
\end{aligned} \tag{14}$$

Only when setting

$$\begin{aligned}
&c = c(z,t), \quad d = d(z,t), \quad e = d(z,t), \\
&a(x,z,t) = b_x(x,z,t) - h'(t),
\end{aligned} \tag{15}$$

the resonance at $j = 2$ occurs. Here $'$ means an ordinary derivative with respect to the temporal variable t. This leads to a nonautonomous $(2+1)$-dimensional Burgers equation:

$$\begin{aligned}
&u_t + \{b_x(x,z,t) - h'(t)\}u + b(x,z,t)u_x + c(z,t)u_z \\
&+d(z,t)uu_z + d(z,t)u_x\partial_x^{-1}u_z \\
&+2\exp\{h(t)\}d(z,t)u_{xz} + g(x,z,t) = 0.
\end{aligned} \tag{16}$$

From the arbitrariness of resonance functions u_1 and u_2, we can set a generalized Cole-Hopf transformation:

$$u = u_0\phi^{-1} = 2\exp\big\{h(t)\big\}\frac{\phi_x}{\phi}. \tag{17}$$

In the case of $g(x,z,t) = 0$, by this transformation, equation (16) is reduced to a linear equation,

$$\begin{aligned}
&\phi_t + b(x,z,t)\phi_x + c(z,t)\phi_z \\
&+2\exp\big\{h(t)\big\}d(z,t)\phi_{xz} = 0.
\end{aligned} \tag{18}$$

Setting $z = x$, the nonautonomous $(2+1)$-dimensional Burgers equation (16) is reduced to one in reference [5], which demonstrates that nonautonomous lower dimensional Burgers equation can be reduced to the autonomous Burgers equation if terms satisfy a compatibility condition.

2.3. Nonautonomous KdV type equations

We discuss a following higher dimensional KdV type equation for $u = u(x,z,t)$:

$$\begin{aligned}
&u_t + a(x,z,t)u + b(x,z,t)u_x + c(x,z,t)u_z \\
&+d(x,z,t)uu_z + e(x,z,t)u_x\partial_x^{-1}u_z \\
&+f(x,z,t)u_{xxz} + g(x,z,t) = 0.
\end{aligned} \tag{19}$$

Equation (19) includes the standard higher dimensional KdV one, so-called Calogero-Bogoyavlenskii-Schiff (CBS) equation[17]:

$$u_t + uu_z + \frac{1}{2}u_x\partial_x^{-1}u_z + \frac{1}{4}u_{xxz} = 0, \tag{20}$$

which, by setting $z = x$, reads the (ordinary) KdV equation which is well-known to be integrable. We determine the coefficients of (19) to pass the Painlevé test. Here a potential field $U = U(x,x,t)$ for the original one u is defined as

$$u = U_x, \tag{21}$$

since the non-local term of equation (19) should eliminate to perform the Painlevé test. Then we are now looking for a solution of equation (19) in terms of U in the Laurent series expansion:

$$U = \phi^\alpha \sum_{j=0}^{\infty} U_j\phi^j, \tag{22}$$

where U_j are analytic functions of the independent variables in a neighborhood of $\phi = 0$. In this case, leading order is -1 and

$$U_0 = \frac{12f(x,z,t)}{d(x,z,t) + e(x,z,t)}\phi_x, \tag{23}$$

is given. Then, it is found that the resonance occur at

$$j = -1, 1, 4, 6, \tag{24}$$

substituting the expansion (22) with $\alpha = -1$ into equation (19) in terms of U. We are succeeded in finding four types of the nonautonomous higher dimensional KdV equation. We shall display only two types, one of them is

$$\begin{aligned}
&u_t + \frac{2}{3}x\big\{\alpha(z,t) - \beta(t) + c_z(z,t)\big\}u_x + c(z,t)u_z \\
&+\left(\frac{d'(t)}{d(t)} - \frac{f'(t)}{f(t)} + \frac{4}{3}\big\{\alpha(z,t) - \beta(t) + c_z(z,t)\big\}\right)u \\
&+d(t)uu_z + \frac{d(t)}{2}u_x\partial_x^{-1}u_z + f(t)u_{xxz} + g(z,t) \\
&= 0,
\end{aligned} \tag{25}$$

and another is

$$u_t + \left(2B_1(z,t) - \eta'(t)\right)u + c(z,t)u_z + \{B_1(z,t)x$$
$$+B_2(z,t)\}u_x + d(z,t)uu_z + \frac{d(z,t)}{2}u_x\partial_x^{-1}u_z$$
$$+\frac{3}{2}\exp\{\eta(t)\}d(z,t)u_{xxz} + g(z,t) = 0, \qquad (26)$$

where $\alpha(z,t)$, $\beta(t)$, $B_1(z,t)$, $B_2(z,t)$, $\eta(t)$ are arbitrary functions. We note that the rate of coefficients of uu_z and $u_x\partial_x^{-1}u_z$ are just 2, this magic number is shared by equation (25), (26) and also the CBS equation (20). Setting $z = x$, the nonautonomous $(2+1)$-dimensional KdV equations (25) and (26) are reduced to ones in reference [5, 8, 9].

3. Conclusions

In this manuscript, we have presented new nonautonomous $(1+1)$- and $(2+1)$- dimensional integrable equations. In section 2 we have reviewed the Painlevé test and nonautonomous higher dimensional Burgers equations is constructed. Via truncating the Laurent expansion, we have seen the generalized Cole-Hopf transformation. And then nonautonomous higher dimensional KdV equations have been also obtained. In reference [18], exact solutions, hierarchies and families of nonautonomous higher dimensional equations (16), (25) and (26) are reported in details.

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