2004 International Symposium on Nonlinear
Theory and its Applications (NOLTA2004)
Fukuoka, Japan, Nov. 29 - Dec. 3, 2004

Existence of Horseshoes in Current-Mode Controlled Boost Converters

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Abstract— In this paper, a current-mode controlled boost converter is studied and the existence of chaos is proven theoretically in this system. The proof consists of showing that the dynamics of the system is semiconjugate to that of a shift map, which implies positive entropy of the system and hence chaotic behavior. The essential tool is the horseshoe hypotheses proposed by Kennedy and Yorke, which will be reviewed prior to the discussion of the main finding.

Keywords— Proof of chaos, switching converters, horseshoe map.

1. Introduction

In the last decade, the study of nonlinear phenomena in switching power converters has attracted much attention from both the engineering and the nonlinear science communities [1, 2, 3]. It has been shown that switching power converters can exhibit a variety of complex behavior such as bifurcation and chaos, as a result of the switching action. Although chaotic behavior has been widely recognized in switching power converter, no rigorous proof of chaos in such systems has been reported. Up to now, only a discussion of the existence of a Smale horseshoe mechanism in a voltage-mode controlled buck converter was reported by Olivar *et al.* [4]. The purpose of this paper is to provide a rigorous proof of chaos in a switching power converter. This is realized by proving the existence of a horseshoe in the describing Poincaré map. The proof is based on the work of Kennedy and Yorke [5], which gives a set of sufficient conditions for a continuous map to exhibit chaotic behavior.

2. Boost Switching Converter Under Current-Mode Control

The system studied here is a current-mode controlled boost converter shown in Fig. 1, which has been found to be chaos prone by simulation and experimental studies [6, 7, 8]. The state variables for this system are the inductor current, i, and the output voltage, v. For simplicity, we consider continuous-conduction-mode (CCM) operation, in which the inductor current

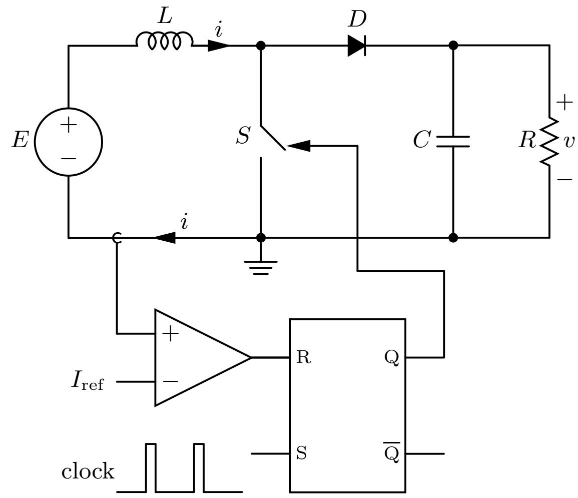

Figure 1: Current-mode controlled boost converter.

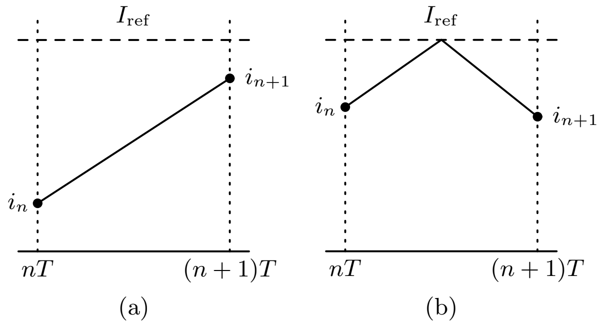

Figure 2: Typical inductor current waveforms (a) for $I_{\rm ref} - i_n \geq ET/L$; and (b) for $I_{\rm ref} - i_n < ET/L$.

never falls to zero. Suppose we define the Poincaré map as an iterative function that expresses the state variables at the end of a switching period in terms of those at the beginning of the period, i.e.,

$$x_{\rm n+1} = f(x_n, \mu) \tag{1}$$

where x_n is a vector consisting of the state variables at $t = nT$, T being the switching period, i.e.,

$$x_n = \begin{pmatrix} i_n \\ v_n \end{pmatrix} = \begin{pmatrix} i(nT) \\ v(nT) \end{pmatrix}. \tag{2}$$

Normally the operation can be described as follows.

Suppose the period begins with the switch turned on. The inductor current rises linearly, whereas the output voltage decays exponentially. As soon as the inductor current reaches a reference level, $I_{\rm ref}$, the switch is turned off. The inductor current thus falls, since the output voltage is much higher than the input voltage under normal operating condition. At the end of the period, the switch is turned back on, and the cycle repeats. Typically, the inductor current waveform is shown in Fig. 2. Clearly, depending on the value of i at the beginning of the period, the switch may or may not turn off during a switching period. Specifically, provided $I_{\rm ref} - i_n < ET/L$, the operation is as described above, with the switch turned off once during a period, as illustrated in Fig. 2 (b). However, if $I_{\rm ref} - i_n \geq ET/L$, the switch remains on for the whole period, as shown in Fig. 2 (a). Thus, the corresponding Poincaré map can be written as follows [6].

For the case $I_{\rm ref} - i_n \geq \frac{ET}{L}$, the Poincaré map is

$$\begin{pmatrix} i_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} i_n + \frac{ET}{L} \\ v_n e^{-2kT} \end{pmatrix} \tag{3}$$

and for the case $I_{\rm ref} - i_n < \frac{ET}{L}$, it is

$$\begin{pmatrix} i_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} e^{-k\tau}(A_1 \sin(\omega\tau) + A_2 \cos(\omega\tau)) + \frac{E}{R} \\ E - \frac{e^{-k\tau}}{\omega}\left[(kv'_n - kE - \frac{A_2}{C})\sin(\omega\tau) \right. \\ \left. + (E - v'_n)\cos(\omega\tau)\right] \end{pmatrix} \tag{4}$$

where

$$k = \frac{1}{2RC}, \tag{5}$$

$$\omega = \sqrt{\frac{1}{LC} - k^2}, \tag{6}$$

$$\tau = T - \frac{L}{E}(I_{\rm ref} - i_n), \tag{7}$$

$$v'_n = v_n e^{-2k(T-\tau)}, \tag{8}$$

$$A_2 = I_{\rm ref} - \frac{E}{R}, \tag{9}$$

$$A_1 = \frac{kLA_2 + E - v'_n}{\omega L}. \tag{10}$$

Finally, as an operational requirement for CCM, we need to ensure a sufficiently large inductance or short period. For example, we may choose $T = 100$ μs, $R = 20$ Ω, $L = 1.5$ mH, $C = 12$ μF, $E = 10$ V and $I_{\rm ref} = 3.5$ A. A chaotic attractor from this Poincaré map is shown in Fig. 3.

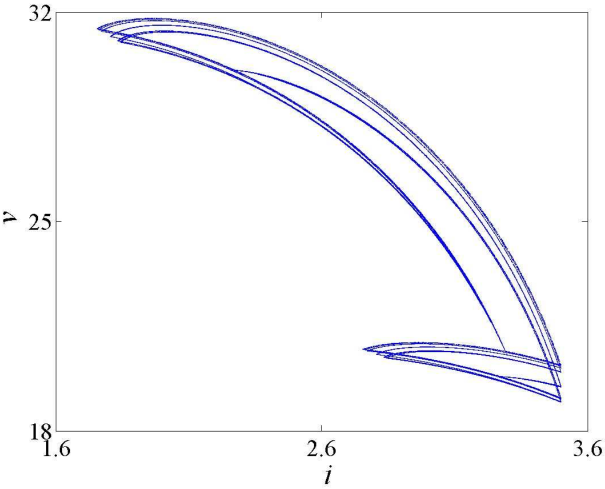

Figure 3: The chaotic attractor of current-mode controlled boost converter.

3. Review of Metric Space and the Horseshoe Hypotheses

In this section, we begin with a review of a fundamental result regarding the horseshoe dynamics [9], which is essential to proving the existence of a horseshoe in the current-mode controlled boost converter.

Let $S = \{1, 2, 3, ..., N\}$ be a collection of symbols. Then, a bi-infinite sequence can be constructed with all elements from S. Let Σ^N be the space of all bi-infinite sequences using the symbol set S. Any point s in Σ^N can be written as

$$s = \{\cdots s_{-n} \cdots s_{-1}.s_0 s_1 \cdots s_n \cdots\} \tag{11}$$

where $s_i \in S$ for any i. In order to describe limit processes in Σ^N, it is convenient to define a metric on Σ^N. Consider another point $\bar{s} = \{\cdots \bar{s}_{-n} \cdots \bar{s}_{-1}.\bar{s}_0 \bar{s}_1 \cdots \bar{s}_n \cdots\}$ in Σ^N. The distance between s and $\bar{s}$ is defined as

$$d(s, \bar{s}) = \sum_{i=-\infty}^{\infty} \frac{1}{2^{|i|}} \frac{d_i(s_i, \bar{s}_i)}{1 + d_i(s_i, \bar{s}_i)}, \tag{12}$$

where

$$d_i(s_i, \bar{s}_i) = \begin{cases} 1 & \text{if } s_i \neq \bar{s}_i, \\ 0 & \text{if } s_i = \bar{s}_i. \end{cases} \tag{13}$$

With the distance defined above, the space Σ^N is a *metric space* with the metric $d(\cdot, \cdot)$. In the following, we will state some results concerning the structure of Σ^N without proofs. Readers may refer to Wiggins [9] for detailed proofs.

Proposition 1 *The space Σ^N equipped with the metric defined by Eq. (12) is compact, totally disconnected, and perfect.*

The above proposition essentially states the properties that define the structure of the metric space Σ^N. It may be worth noting that compactness, total disconnectedness and perfectness are often taken as the defining properties of a *Cantor set*, which is used in the characterization of complex structures of invariant sets in a chaotic dynamical system [9].

Proposition 2 *Consider a shift map $\sigma : \Sigma^N \to \Sigma^N$, which is defined as*

$$\sigma(s) = \{\cdots s_{-n} \cdots s_{-1} s_0 . s_1 \cdots s_n \cdots\} \quad (14)$$

for $s = \{\cdots s_{-n} \cdots s_{-1} . s_0 s_1 \cdots s_n \cdots\} \in \Sigma^N$. The following statements regarding this shift map are true:

1. *The metric space Σ^N is invariant over σ, i.e., $\sigma(\Sigma^N) = \Sigma^N$.*
2. *σ is continuous.*
3. *σ has*
   - *(a) a countable infinity of periodic orbits consisting of orbits of all periods;*
   - *(b) an uncountable infinity of nonperiodic orbits; and*
   - *(c) a dense orbit.*
4. *Σ^N is a chaotic, compact invariant set for σ.*

We remark here that the afore-defined shift map can be equivalently represented by $[\sigma(s)]_i = s_{i+1}$.

Next, we will recall some aspects of the topological horseshoe theory proposed by Kennedy and Yorke [5], which are relevant to the main result to be presented in this paper. Essentially, Kennedy and Yorke found a set of sufficient conditions for a continuous map f to exhibit chaos. These conditions are summarized in what has been called the **horseshoe hypotheses,** which include the following:

1. X is a separable metric space;
2. $Q \subset X$ is locally connected and compact;
3. The map $f : Q \to X$ is continuous;
4. The sets $\mathbf{end}_0 \subset Q$ and $\mathbf{end}_1 \subset Q$ are disjoint and compact, and each component of Q intersects both $\mathbf{end}_0$ and $\mathbf{end}_1$;
5. Q has crossing number $N \geq 2$.

The crossing number N in the above hypotheses can be reasoned as follows. A *connection* Γ is a compact connected subset of Q that intersects both sets $\mathbf{end}_0$ and $\mathbf{end}_1$. A *preconnection* γ is a compact connected subset of Q for which $f(\gamma)$ is a connection. Then, the crossing number N is defined as the largest number such that every connection contains at least N mutually disjoint preconnections.

The main result from Kennedy and Yorke can be summarized in the following theorem [5]:

Kennedy-Yorke's Theorem *Consider a map f. Suppose the horseshoe hypotheses are satisfied. Then, there is a closed invariant set $\Lambda \subset Q$ for which $f|\Lambda$ is semiconjugate to a shift map on N symbols.*[1]

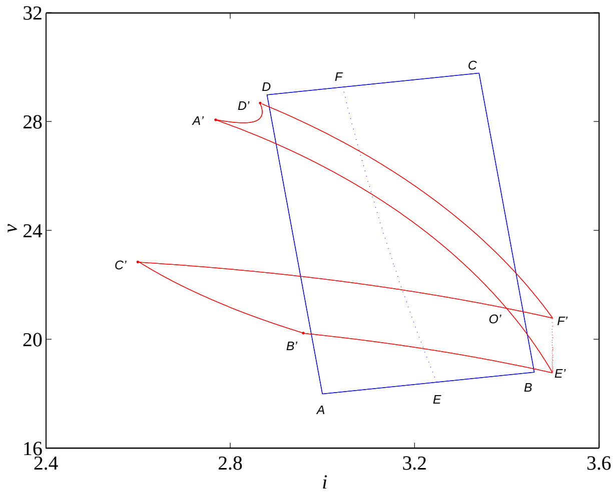

Figure 4: $ABCD$ and its image $A'B'C'D'$ under P.

The main consequence of this result is that the entropy of f is not less than that of σ, which implies that f has positive entropy and is therefore chaotic.

4. Existence of a Horseshoe Map in Current-mode Controlled Boost Converter: A Proof of Chaos

The Poincaré map for the current-mode controlled boost converter has been given in (3) and (4). Denote this map by p. Here, we consider two iterates of this map, i.e., p^2.

For brevity, we denote p^2 by P. Consider the parallelogram $ABCD$ with the coordinates of four endpoints defined as $A = (3, 18)$, $B = (3.46, 18.8)$, $C = (2.88, 29)$ and $D = (3.34, 29.8)$. Figure 4 shows the parallelogram $ABCD$ and its image $A'B'C'D'$ under P. Here, $A' = P(A)$, $B' = P(B)$, $C' = P(C)$ and $D' = P(D)$. The image $A'B'C'D'$ is divided into two parts by the curve $E'F'$, i.e., $A'B'C'D' = A'E'F'D' \cup B'E'F'C'$, where $A'E'F'D' = P(AEFD)$, $B'E'F'C' = P(BEFC)$ and $E'F' = P(EF)$. It is noted that $A'E'F'D'$ intersects with $B'E'F'C'$ and $E'F' \subset O'E'F' = A'E'F'D' \cap B'E'F'C'$, which implies that P is not a homeomorphism.

Now, we choose the line AD and BC as the two end subsets required in the horseshoe hypotheses. To ease the construction of the preconnections, we look at $ABCD$ and its image $A'B'C'D'$ again. It is straightforward to find that $A'E'F'D' \cap AD = GH$ and $A'E'F'D' \cap BC = IJ$. Similarly, $B'E'F'C' \cap BC =$

[1] A shift map on N symbols is equivalent to $\sigma|\Sigma^N$. Also, if $f|\Lambda$ is semiconjugate to $\sigma|\Sigma^N$, there exists a continuous and onto map $\phi : \Lambda \to \Sigma^N$ such that $\phi \circ f = \sigma \circ \phi$.

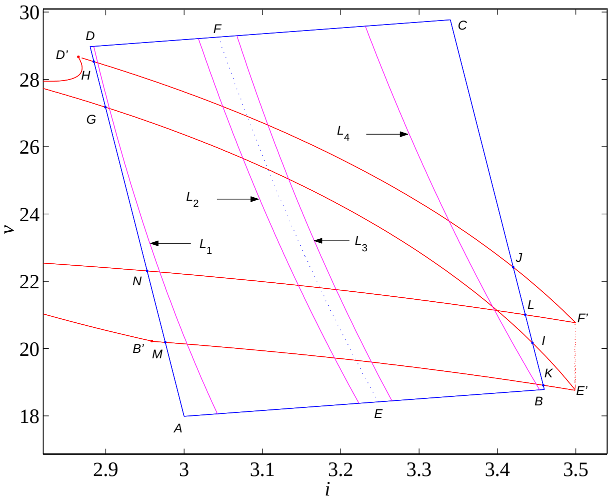

Figure 5: Enlargement of Fig. 4 showing intersections of $A'B'C'D'$ with two ends (AD and BC) and preimages corresponding to these intersections.

KL and $B'E'F'C' \cap AD = MN$. By numerical computation, the corresponding preimages of the lines GH, IJ, KL and MN can also be obtained, i.e., $P(L_1) = GH$, $P(L_2) = IJ$, $P(L_3) = KL$ and $P(L_4) = MN$, as shown in Fig. 5. It should be noted that, in Fig. 5, the zone encompassed by L_1 and L_2 within $ABCD$ (denoted by Ω_1) is disjoint to that by L_3 and L_4 (denoted by Ω_2). It can also be concluded that any curve across the zone Ω_1 (Ω_2) with two ends located on L_1 (L_3) and L_2 (L_4) will be a preconnection. This will be useful for constructing the proof of the following theorem, which summarizes our main finding:

Main Theorem *For the Poincaré map P describing the dynamics of the current-mode controlled boost converter, there exists a closed invariant set $\Lambda \subset Q$, where $Q \subset \mathcal{R}^2$, such that $P|\Lambda$ is semiconjugate to a shift map.*

Proof: Recall the horseshoe hypotheses. First, $\mathcal{R}^2$ (which is X in the horseshoe hypotheses) is separable. Consider $Q = ABCD$ in the afore-described construction. Clearly, $Q \subset X$ is locally connected and compact. The Poincaré map $P : Q \to X$ is continuous. Let $\mathbf{end}_0 = AD$ and $\mathbf{end}_1 = BC$. Then, they are disjoint and compact. The crossing number N can be derived as follows. Referring to the afore-presented construction, given any connection Γ, it will cross both Ω_1 and Ω_2. Moreover, $\Omega_1 \cap \Gamma$ and $\Omega_2 \cap \Gamma$ are two mutually disjoint preconnections according to the previous statement. Thus, the crossing number $N \geq 2$. Hence, the horseshoe hypotheses are satisfied. From Kennedy-Yorke's theorem, there exists a closed invariant set $\Lambda \subset ABCD$ such that $P|\Lambda$ is semiconjugate to a shift map.

5. Conclusion

This paper studies the current-mode controlled boost converter and presents a rigorous proof for the existence of a horseshoe in the describing Poincaré map with the assistance of numerical computation. Our work is based on the result of Kennedy and Yorke [5], which provides a set of sufficient conditions for guaranteeing semiconjugation of a continuous map to the shift map, which then implies chaotic behavior of the continuous map.

Acknowledgment

This work is supported by the Hong Kong Research Grant Council under a Competitive Earmarked Research Grant (No. PolyU 5241/03E).

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