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Algorithm for Convergence to Superstable Periodic Solutions of the Logistic Map

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Abstract—We propose an algorithm by which the parameter of the logistic map is converged from a chaotic initial value to a superstable one in a periodic window. We further improve it to enhance its robustness against noise. Algorithms like this can be applied to memory when regarding superstable orbits as memory states with noise.

1. Introduction

The one-dimensional unimodal map has been extensively studied as the simplest nonlinear map with complex chaotic dynamics [1-3]. A typical example is the logistic map, which has a parameter controlling its nonlinear dynamics. The parameter is generally treated as a constant in time. In some studies, however, this is not constant but changes over time due to state variables. These studies used such configurations to obtain the required parameter value.

For instance, Melby *et al.* [4] proposed a model where the parameter of the logistic map was changed by a low-pass filtered feedback from the time series at every certain steps whose lengths had previously been determined. This means that the control parameter changes more slowly than the variable, originally proposed by Haken [5]. Consequently, the control parameter left the chaotic regime and entered into the periodic window and converged.

Melby *et al.* argued the relation between their model and living things from the viewpoint of adaptation to the edge of chaos, which is the bound between a periodic and a chaotic phase. Suzuki and Kaneko [6] took the logistic map as the dynamics of bird song. As a result of imitation, the dynamics of song, i.e., the parameter value also converges to the edge of chaos.

In this paper, we propose another model for the logistic map that changes the parameter through its own system. Roughly speaking, the map changes its parameter values by setting the maximum of certain interval of the time series to the extremum of the next map. This mechanism causes the parameter value to decrease and converge to the point where the extremum is equal to the maximum. This converged parameter generates a superstable and periodic solution. Hereinafter, we explain the algorithm in details responsible for this process.

2. The algorithm

Consider an one-dimensional unimodal map on one parameter,

$$x(n+1) = f(x(n), a). \tag{1}$$

Here f is a smooth function $f : I \to I, I = [0, 1], f(0, a) = f(1, a) = 0$. Since f is unimodal, it has one critical point x^* in I. As x^* is a solution to $f'(x, a) = 0$, it is written as a function of a, namely

$$x^* = g(a). \tag{2}$$

We now introduce supertrack function $\phi_p(a)$, which is the pth iterate of the critical point [7]. According to $\phi_0(a) = x^*$ and Eq. (2), the supertrack function is defined by

$$\phi_0(a) = g(a) \tag{3}$$

and

$$\phi_p(a) = f(\phi_{p-1}(a), a), \quad p \geq 1. \tag{4}$$

This definition leads to $\phi_1(a)$ being the extremum of $f(x, a)$. If there is a solution to the equation

$$\phi_p(a) = x^* \tag{5}$$

, which is written as a_{sp}, the set $\{g(a_{sp}), \phi_1(a_{sp}), \ldots, \phi_{p-1}(a_{sp})\}$ is called superstable p-periodic solutions. By solving Eq. (5), we obtain the value for a parameter for superstable p-periodic solutions. We call this the superstable p-periodic parameter.

Let us now introduce the algorithm for the map we previously described.

1. Give the initial conditions for the parameter and the variable, $a(0)$, $x(0)$
2. Iterate function f for T steps from $x(0)$ using $a(0)$.
3. Set $\hat{x}$ to the maximum value of time series $\{x(n)\}_{1 \leq n \leq T}$.
4. Solve equation $\hat{x} = \phi_1(a) (\equiv f(g(a), a))$ with respect to a and set the solution as $a(1)$.

5. Return to step 2 and set initial conditions $x(T),\ a(1)$.

Here, we consider a case where there is only one solution to the equation in step 4, i.e. , $\phi_1(a)$ has an inverse function. The logistic map typically satisfies this condition. We apply this algorithm to the logistic map and find that the parameter converges from chaotic initial value to superstable one a_{sp} in a periodic window and analyze the stability of a_{sp}, qualitatively.

2.1. Application of algorithm to logistic map

The logistic map is defined as follows.

$$x(n+1) = ax(n)(1-x(n)),\ a \in [0,4],\ x \in [0,1]. \tag{6}$$

Here, x is a variable and a is a parameter.

The behavior of the logistic map is determined by parameter a. If $0 < a < 3.569\ldots$ the dynamics of x is periodic. However, if $3.569\ldots < a < 4$, the dynamics of x is mostly chaotic. There are, however, values for the parameter in this range that give periodic orbits. These values are called periodic windows. We know that there are thousands of periodic windows. In addition, as there also exists a superstable periodic orbit in parameter values which generate a stable periodic solution in a window, there are too many superstable periodic solutions in the chaotic regime $3.569\ldots < a < 4$.

The dynamical system that represents the algorithm applied to the logistic map is defined as follows.

$$x(n+1) = a(k)x(n)(1-x(n)), \tag{7}$$
$$a(k) = 4\hat{x}_k, \tag{8}$$
$$\hat{x}_k = \max_{0\le j\le T} x(\tau_k - j). \tag{9}$$

Here, $\tau_k = kT,\ k = 0,1,2,\ldots,\ k = \lfloor n/T \rfloor$, where $\lfloor y \rfloor$ is a maximum integer less than y. The extremum $\phi_1(a) = a/4$. Now, as parameter $a(k)$ is no longer constant with time, we call $a(k)$ a nonlinearity variable from now. We also explain the dynamics of nonlinearity variable $a(k)$ qualitatively.

2.1.1. Nonincreasing dynamics of nonlinearity variable $a(k)$

The main process in the algorithm is step 4. This means that the nonlinearity variable is determined such that maximum $\hat{x}_k$ is equal to the extremum of the new return map, namely $\hat{x}_k = \phi_1(a(k+1))$ (property (A)). Since the logistic map is unimodal and T is finite, the values that x can take are less than the extremum, namely, $\hat{x}_k \le \phi_1(a) = a/4$. Here, because of property (A), if $\hat{x}_k < a/4$, then $a(k+1) < a(k)$ and if $\hat{x}_k = a/4$, then $a(k+1) = a(k)$. In the latter, if nonlinearity variable $a(k)$ generates chaotic dynamics in x, according to the nonperiodicity of chaos, the time series does not contain an extremum in the next procedure of the algorithm, that is, $\hat{x}_{k+1} < a/4$. Therefore, the next nonlinearity variable is smaller than the previous, i.e. , $a(k+2) < a(k+1)$. However, if $a(k)$ generates periodic dynamics, the maximum is also the same as the extremum, that is, $\hat{x}_{k+1} = a/4$ in the next procedure. Therefore, the next nonlinearity variable is also the same as the previous, i.e. , $a(k+2) = a(k+1)$. This means that $a(k)$ converges fixed point $\bar{a}$ in this case. From the definition of a superstable periodic parameter, $\bar{a}$ is equivalent to a_{sp}. Thus, we can see that the dynamics of the nonlinearity variable is nonincreasing.

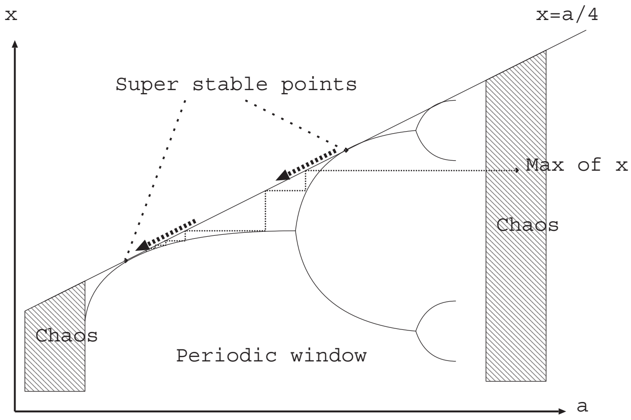

Figure 1: Stability of superstable (fixed) point in (a, x)-plane. Dashed line indicates dynamics of nonlinearity variable.

2.1.2. Stability of fixed point $\bar{a}$

As previously discussed, a fixed point is a superstable periodic parameter. Furthermore, there is a superstable p-periodic orbit in parameter values which generate stable p periodic solutions in a periodic window. Hence, we consider the (a, x)-plane so that we can observe the dynamics of $a(k)$ in a periodic window. Figure 1 shows the dynamics of $a(k)$ in a periodic window. Line $\phi_1(a) = a/4$ is the track of the extremum. Since the maximum values of periodic solutions are less than the extremum values, the fixed point is a tangent between $\phi_1(a) = a/4$ and the track of the maximum solution. In the figure, step 4 of the algorithm is equivalent where $a(k+1)$ is the intersection of $x = a/4$ and the horizontal line from $(a(k), \hat{x}_k)$ in the (a, x)-plane. This process is indicated in Figure 1 as a dashed line. By iterating this process, the nonlinearity variable converges to the tangent, that is, superstable periodic parameter $a_{sp} = \bar{a}$. Looking at Figure 2, if nonlinearity variable $a(k)$ is less than $\bar{a}$, then $a(k)$ diverges from $\bar{a}$. This ensures that $a(k)$ is stable at $\bar{a} < a(k)$ and unstable at $a(k) < \bar{a}$, considering perturbation around $\bar{a}$. Therefore, we can see that the fixed point is like a Milnor attractor. As we describe later, this stability in the algorithm is not effective against noise.

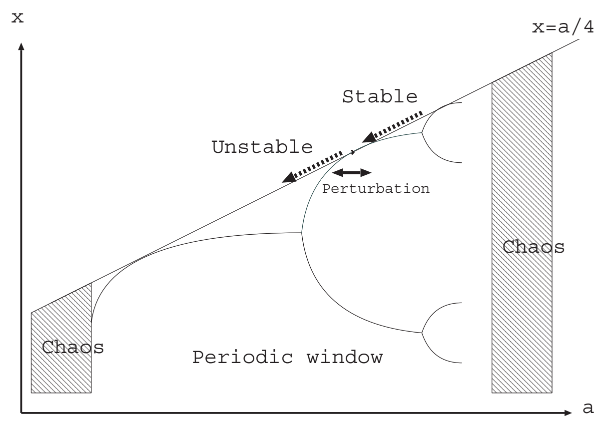

Figure 2: Stability with noise of superstable point in (a, x)-plane.

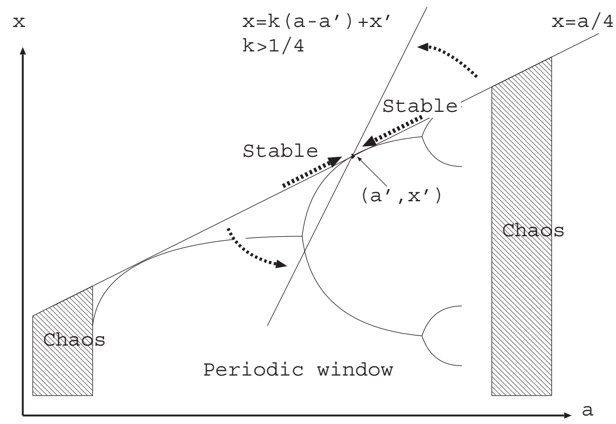

Figure 3: Improved algorithm in (a, x)-plane.

3. Improvements in adding noise

The algorithm cannot accommodate for noise because of the following. 1) Fixed point $\bar{a}$ is not asymptotically stable in the dynamics of $a(k)$ (Cf. 2.1.2). Therefore, if nonlinearity variable $a(k)$ is perturbed from the fixed point toward $a(k) < \bar{a}$, it never returns to $\bar{a}$. 2) According to the algorithm, noise added to variable x also have an effect on nonlinearity variable $a(k)$. These two facts imply that the fixed point dose not retain its state stably when noise is added.

To avoid this, we improve the algorithm as follows. Assume that noise is not added constantly but periodically and the superstable parameter is given by the conventional algorithm without noise. Figure 3 shows the improved algorithm where the line $x = a/4$ is rotated steeply to focus on the superstable point (a_{sp}, x'). That is, the new line is $x = k(a - a_{sp}) + x'$, $k > 1/4$. This is called the criterial line, which does not cross the track of the maximum solution except the targeting superstable point. This procedure can make the superstable point asymptotically stable in terms of the dynamics of the nonlinearity variable $a(k)$. Furthermore, the dynamics of variable x is superstable at that point.

4. Numerical results

We do numerical simulations of the algorithm and apply these to the logistic map. Table 1 lists $\bar{a}$, period, and its corresponding parameter value calculated by solving eq. (5) with Newton method for several different values of $x(0)$ and $a(0) = 4$. We determine the period to an accuracy of 10^{-6} and the periodicity is determined by negative Lyapunov exponents. In every value of $x(0)$, the nonlinearity variable successfully converges to a superstable parameter value.

Figure 4 shows the time series of nonlinearity variable $a(k)$. Initial value of nonlinearity variable $a(0)$ is also 4. As discussed in subsubsection 2.1.1, a decreases and converges to superstable periodic parameter a_{sp1} (corresponding period: 24). Moreover, as discussed in subsubsection 2.1.2, negative small perturbation (-10^{-6}) is added at step 500 changes a_{sp1} to smaller superstable periodic parameter a_{sp2} (corresponding period: 18).

Figure 5 shows the time series when noise is added. In the first 30,000 steps, we obtained a superstable parameter with the original algorithm. After that, we use the improved algorithm and add noise every 15,000 steps. The noise follows normal distribution $N(x,\ 0.01)$. The slope of the criterial line is 0.5. The results reveal that our improved algorithm can maintain a superstable 18 periodic solution robustly against noise.

Table 1: Relation between $x(0)$, period, and $\bar{a}$. $T = 100$. Bottom row lists superstable parameter values calculated by solving Eq. (5) with Newton method. $a(0) = 4$.

$x(0)$	0.1	0.2	0.3	0.4
$\bar{a}$	3.85210	3.96093	3.85033	3.85511
period	15	8	18	24
a(Eq. (5))	3.85210	3.96093	3.85033	3.85511

5. Discussion

As we can see from the numerical results, the period generated by the converged superstable parameter is not predictable. This is because the nonlinearity variable goes through chaotic regimes, so that the dynamics of variable x is not predictable. Therefore, it is impossible to predict the period generated by the converged parameter from the initial condition. So we suggest another usage for the algorithm, especially the improved one.

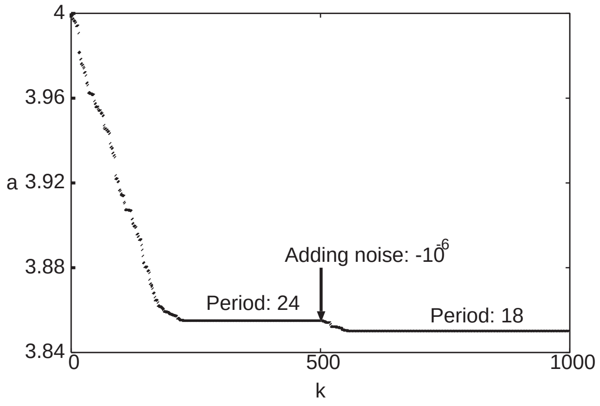

Figure 4: Time series for nonlinearity variable $a(k)$. Initial conditions are given as $a(0) = 4$ and $x(0) = 0.4$. First $a(k)$ converges to superstable parameter value, which generates period 24 solution. Then, adding noise at $k = 500$ shifts orbit to period 18.

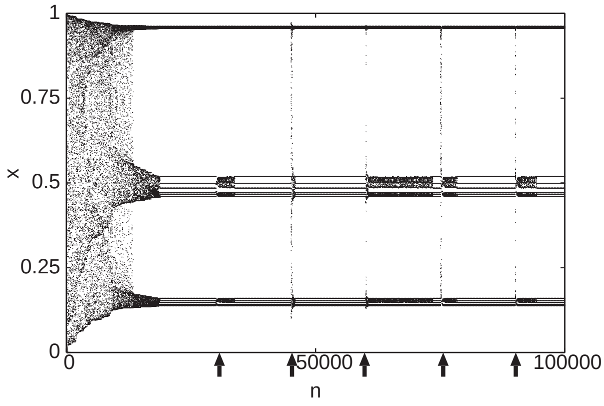

Figure 5: Time series for x when noise is added. In the first 30,000 steps, the original algorithm is applied, after which the improved algorithm is applied. Noise is added every 15,000 steps starting from 30,000 as indicated by the arrows. $k = 0.5$. Period 18 is maintained.

As previously discussed, there are a numerous number of superstable periodic solutions in the chaotic regime $3.569\ldots < a < 4$. The orbits generated by these parameter values are attractors. Therefore, if each attractor corresponds to some information, we can employ these to store data as memory states. If this memory is implemented, one device represented by the dynamics of the logistic map has many separate data storage states and these states can be maintained superstably.

Our algorithm is efficient because the parameter value is maintained dynamically by the interaction of variables and nonlinearity variables. To apply the algorithm to memory where the period represents information for instance, we need to consider the following two steps.

1. Solve Eq. (5) that obtains a superstable periodic solution with the required period.

2. Maintain the parameter with the improved algorithm.

Here, step 1 is interpreted as a static process, while step 2 is dynamic. These steps imply that superstable periodic orbits can be employed for superstable memory states.

In conclusion, the algorithm we have proposed converges the parameter from a chaotic initial value to a superstable periodic one. If the process leading to a superstable periodic parameter value is taken as a data storage state, the improved algorithm against noise can be applied to superstable memory. However, it needs a quantitative analysis on the number of conservable states and the relation between these and the effect of noise with respect to the numerical precision.

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