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Generation of Optimal Constrained Switching Pattern for Single-Phase Inverter

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Abstract— In this paper, we consider a new methodology to obtain a Pulse Width Modulated (PWM) signal encoding an over-sampled sinusoidal waveform for single-phase inverter drives. More specifically, the obtained switching pattern is optimum in the sense that it minimizes the spurious harmonic power within a certain band when subject to two constraints: a finite time-resolution and a maximum switching frequency. The first constraint is necessary when the inverter drive control is implemented by a digital system (i.e. a micro-controller unit), while the second one assures a minimum duration of each phase of the driving signal.

Due to these constraints, traditional analysis tools for PWM control technique cannot be applied. We will give a detailed description of the novel encoding scheme, and we will compare its performance, in terms of spurious harmonic power, with the ones obtained with traditional modulation schemes based on the methodology reported in [1].

1. Introduction

It is generally recognized [2] that optimized PWM switching pattern computations can offer a significant improvement in the performance of single-phase full bridge inverters, used in traditional power conversion schemes. Programmed PWM techniques optimize a particular objective function in order to obtain harmonics minimization or elimination [3]. It is worth noting, that the objective function is chosen to generate an optimal switching pattern, which minimizes unwanted harmonic effects (due to the switching process) at the inverter output. To cope with the difficulties of calculating the optimal switching instants, the optimized PWM patterns are commonly developed off-line [1, 4], by applying minimization techniques to a set of non-linear equations. The optimal switching pattern are then stored in a non-volatile memory to preprogram a micro-controller unit, which is used for on-line generation of the PWM at real time.

One of the main problems of these approaches is that the sequence generation rely on an infinite time resolution, i.e. it is assumed hat every switching instant can be set with infinite precision along the time axis This is obviously not achievable in any practical application. As an example, in motor control systems the whole control circuitry is implemented by a micro-controller core, which, generally, has only a finite time resolution equal, at best, to its system clock period. The paper is organized as follows: in section 2 the classical methods of harmonic elimination are reviewed, in section 3 the spectral effects of finite time resolution over these methods are analytically evaluated and a numerical example is presented. Finally, in section 4 a new approach to find an optimal switching pattern is presented and a numerical example that shows the spectral improvement over classical methods is also reported.

2. The Classical Solutions

Due to the high industrial relevance, many researchers attempt to give the optimal solution for single phase drives switching pattern. The original idea is to fix the exact number of switch n per sinusoidal period and to find, through Fourier analysis and solving a set of nonlinear equations, the optimal position of the switching angles α_i for each transition. The quarter wave symmetry of the PWM waveform is also assumed for the switching angles, i.e.

$$\alpha_i = \begin{cases} \alpha_i & \text{if } 0 \leq i \leq n/4-1 \\ \pi - \alpha_{n/2-1-i} & \text{if } n/4 \leq i \leq n/2-1 \\ \pi + \alpha_{i-n/2} & \text{if } n/2 \leq i \leq 3n/4-1 \\ 2\pi - \alpha_{n-1-i} & \text{if } 3n/4 \leq i \leq n-1 \end{cases} \tag{1}$$

The flavor of these methods is as follows. First, the switching sequence, that is assumed to be periodic with period T_s, is expressed into Fourier series. Due to the constraint (1) the Fourier series of the switching pattern contains only the $\sin(\cdot)$ terms and all the even harmonics are null. With this, $s(t)$ could be written as

$$s(t) = \sum_{k=0}^{\infty} s_{2k+1} \sin(2\pi k f_s t) \tag{2}$$

where the coefficients s_k may be obtained through Fourier analysis

$$s_k = \begin{cases} 0 & \text{if } k \text{ even} \\ \frac{4}{k\pi}\left(-1 + 2\sum_{i=0}^{n-1}(-1)^i \cos(2\pi k f_s \alpha_i)\right) & \text{if } k \text{ odd} \end{cases} \tag{3}$$

So, considering only the first $n/2$ harmonics, the traditional PWM harmonic elimination techniques are grounded on the possibility to find a suitable combination of $\{\alpha_0, \alpha_1, \ldots, \alpha_{n/4-1}\}$ to force the first odd $n/4-1$ harmonics to vanish, except for the first one that is constrained to a prescribed value. In other words, they attempt to solve the set of $n/4$ nonlinear equations:

$$\begin{cases} \cos(2\pi\alpha_0) - \cos(2\pi\alpha_1) + \ldots - \cos(2\pi\alpha_{n/4-1}) = \frac{A\pi+4}{8} \\ \cos(6\pi\alpha_0) - \cos(6\pi\alpha_1) + \ldots - \cos(6\pi\alpha_{n/4-1}) = \frac{1}{2} \\ \ldots \\ \cos((n-2)\pi\alpha_0) - \cos((n-2)\pi\alpha_1) + \ldots \\ \qquad - \cos((n-2)\pi\alpha_{n/4-1}) = \frac{1}{2} \end{cases} \tag{4}$$

where A is the amplitude of the sinusoidal tone that ideally feeds the motor drive. Several methods have been proposed in the literature to solve (4). To carry out the comparison between traditional harmonic elimination techniques and our algorithm, we will refer to the method proposed in [1], which is based on the computation of the roots of a single univariate polynomial of degree $n/4-1$, and it is very efficient in solving (4) for large n.

2.1. Example

An example of the results obtained by the application of the traditional methods for harmonic elimination is shown in figure 1. The parameters value for the example are: $n = 8$, i.e. the number of switch in a quarter of period, $f_s = 50\text{Hz}$, and $A = 0.6\text{V}$. The switching waveform is shown together with the signal retrieved by means of an ideal low pass filter with cutoff frequency $B = 800\text{Hz}$. As expected, the output signal is a pure sinusoidal tone, with the same amplitude of the encoded one, because all the in-band spurious harmonics are null.

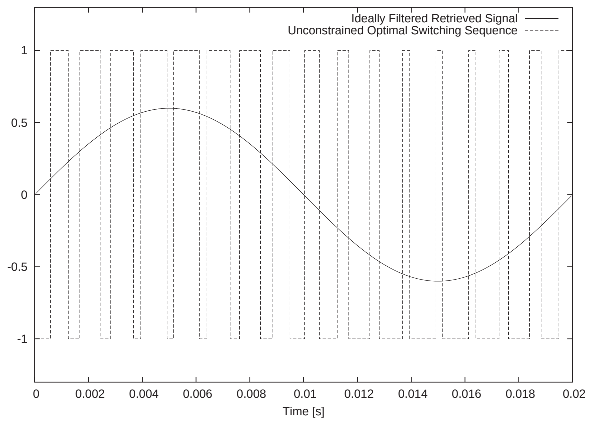

Figure 1: Harmonic Elimination Example.

3. Effects of Finite Time Resolution in PSD

Whenever a two level PWM signal is implemented by a DSP, a problem of temporal resolution arises. In other words, the switching instants of the PWM waveforms are constrained to be at integer multiples of the digital clock period. This situation is highlighted in Figure 2. Note that theoretically, $s(t)$ has no constraint on switching instants, while its digital implementation $u(t)$ has switching instants that are “hooked” to the ones of the digital clock. The vertical arrows represent the time series of the constrained switching process, which is clearly equivalent to periodically sampling $s(t)$ with a period equal to the clock period Δt. The upward arrow represents a “+1” sampled value, while a downward arrow represents “-1” sampled value. Finally, the PWM signal with quantized switching instants $u(t)$ may be obtained by convolving the time series with $g(t)$ the unit pulse of duration Δt. Following the steps suggested by Figure 2, we easily find:

$$u(t) = \sum_{k=-\infty}^{\infty} s\left((k+\frac{1}{2})\Delta t\right) g(t-k\Delta t) \tag{5}$$

Now, we want to quantify the effects of finite time resolution on the power spectrum density of the optimal switching patterns generated by the traditional harmonic elimination schemes. First, if we consider that the two level waveform (2) is written as an

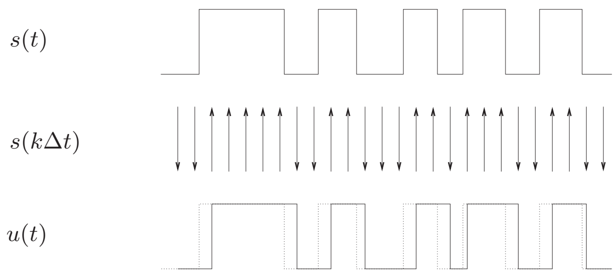

Figure 2: Quantization of switching instants due to finite time resolution.

infinite summation of properly weighted sinusoidal tones, it is straightforward to find the expression of the PSD of $S_s(f)$ [5]. Exploiting the orthogonality of the sinusoidal tones at different frequencies, we obtain:

$$S_s(f) = \sum_{i=-\infty}^{\infty} \frac{|s_i|^2}{4} \delta(f - if_s) \tag{6}$$

where $\delta(\cdot)$ is the Dirac Delta distribution.

We are interested in the relation between $S_s(f)$ and $S_u(f)$, i.e. the PSD corresponding to the deterministic signal $u(t)$. To simplify the analysis we constraint the number of pulses inside a period of the fundamental tone, N, to be an integer. In formulas $N = \frac{1}{f_s \Delta t} \in \mathbb{N}$, where $\mathbb{N}$ is the set of integers. In this case, the deterministic signal $u(t)$ will inherit the periodicity of period $1/f_s$ by the original signal $s(t)$, and will be completely described by the coefficients u_i of its Fourier series. Hence, we have to find the relation between the coefficients s_i in equation (3) and the coefficients u_i. It may be shown [6] that, under the above described conditions, u_i has the following expression

$$u_i = \text{sinc}\left(\pi \frac{i}{N}\right) \frac{1}{N} \sum_{k=0}^{N-1} s\left(\frac{k+1/2}{Nf_s}\right) e^{-\mathbf{i}\, 2\pi \frac{ik}{N}} \tag{7}$$

where $\text{sinc}(\cdot) = \sin(\cdot)/(\cdot)$. Expanding the sampled values in (7) as

$$s\left(\frac{k+1/2}{Nf_s}\right) = \sum_{n=-\infty}^{\infty} \frac{s_n}{2\mathbf{i}} e^{\mathbf{i}\, 2\pi n \frac{i+1/2}{N}} \tag{8}$$

and substituting (8) in (7), after some straightforward calculation we find the relation that links s_i and u_i, i.e.:

$$u_i = \text{sinc}\left(\pi \frac{i}{N}\right) \sum_{h=-\infty}^{\infty} \frac{s_{hN-i}}{2\mathbf{i}} e^{\mathbf{i}\, \pi \frac{hN-i}{N}} \tag{9}$$

where the summation indicates the aliasing folding effect, in agreement with the sampling theory [5]. With this, the PSD of $u(t)$ may be easily calculated as:

$$S_u(f) = \sum_{i=-\infty}^{\infty} |u_i|^2 \cdot \delta(f - if_s) \tag{10}$$

3.1. Example

In this subsection we show an example of the effects of finite time resolutions on the PSD of the retrieved signal at the low pass filter output. We set $n = 8$ (the number of odd eliminated harmonics), $f_s = 50\text{Hz}$, and $A = 0.6\text{V}$. The switching waveform is shown together with the retrieved one by means of an ideal low pass filter with cutoff frequency $B = 800\text{Hz}$, while the time resolution is $\Delta t = 39\mu s$. The time domain waveforms are shown in figure 3, where the in-band harmonic distortion is evident. This is because the out-of-band harmonics, which have been eliminated with the traditional methods, are folded back in-band according to equation (9). To highlight this, the PSD of the waveform in 3 is shown in 4. With this set of parameters, the harmonic distortion in the band B is $539\mu W$.

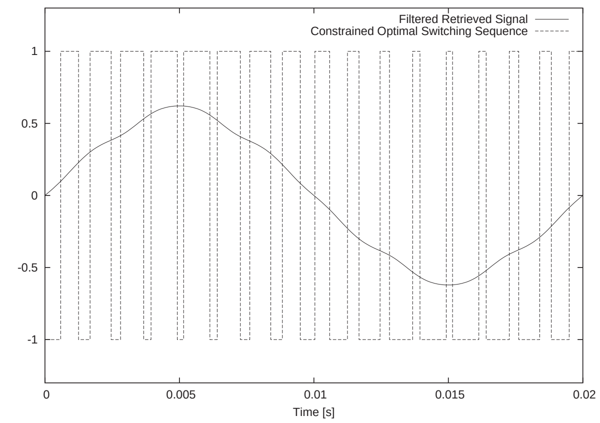

Figure 3: Effects of finite time resolution on the output waveform.

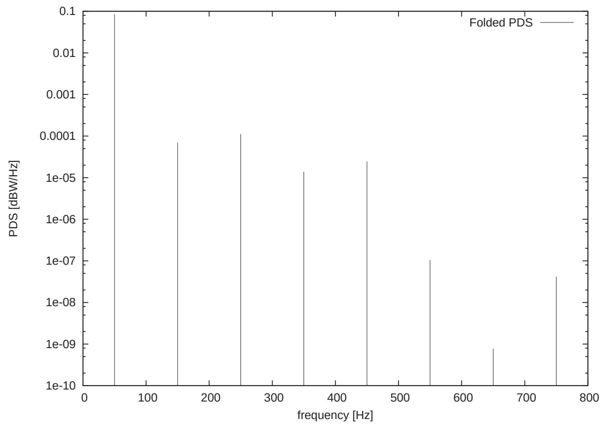

Figure 4: PSD of the finite time resolution waveform.

4. A Different Approach

We want to approximate an analogic sinusoidal signal $f(t)$, that represents a motor drive control signal, with a low-pass filtered version of a two level ($\{-1, +1\}$) signal $s(t)$, that represents the switching control signal of the single-phase inverter drive. To keep into account finite time resolution, its switching instants are assumed to be constrained at integer multiples of Δt, i.e. a suitable multiple of the micro-controller system clock. The ideal signal that should drive the motor is

$$f(t) = A \sin(2\pi f_s t) \tag{11}$$

where A is the amplitude control parameter, and f_s is an assigned frequency (typically $50 \div 60\text{Hz}$ of the electricity grid). The two-level function $s(t)$, whose filtered version will approximate $f(t)$, may be written in the form

$$s(t) = \sum_{i=-\infty}^{\infty} x_i g(t - i\Delta t) \tag{12}$$

where $g(\cdot)$ is the unit pulse of duration Δt and x_h are binary valued $\{-1, +1\}$. Our aim is to find the combination of the coefficients x_i's that assures the best approximation according to a given optimality criterion, that will be exposed below. Furthermore, we introduce a constraint on the ratio between $T_s = \frac{1}{f_s}$ and Δt, i.e. we consider $\frac{T_s}{\Delta t} = N \in \mathbb{N}$. Under this condition, one may easily verify that $s(t)$ inherits the periodicity property from $f(t)$, with the same period T_s. With this, we may consider (12) as the periodic repetition of any piece of $s(t)$ constituted by N symbols, i.e. we may rewrite as $s(t) = \sum_{k=\infty}^{\infty} \sum_{i=0}^{N-1} x_i g(t - i\Delta t - kN\Delta t)$, and, exploiting the periodicity of period $N\Delta t$, we obtain

$$s(t) = \sum_{i=0}^{N-1} x_i \sum_{k=-\infty}^{\infty} g_k e^{\mathbf{i}\, 2\pi k \frac{1}{N\Delta t}(t - i\Delta t)} \tag{13}$$

where $g_k = \frac{1}{N}\text{sinc}(\frac{\pi k}{N}) e^{-\mathbf{i}\, \frac{\pi k}{N}}$.

Now, the switching signal $s(t)$ is low-pass filtered with an ideal filter with cutoff frequency B. Hence, the low-pass filtered output $s_{lp}(t)$ may be written

$$s_{lp}(t) = \sum_{i=0}^{N-1} x_i \sum_{k=-K}^{K} g_k e^{\mathbf{i}\, 2\pi k \frac{1}{N\Delta t}(t - i\Delta t)} \tag{14}$$

where $K = \lfloor \frac{B}{f_s} \rfloor$ keeps only the harmonic content of the signal in the band $[-B, B]$. We note that (14) has a similar form to (13), and the only difference is the inner summation limits.

Let us now define the time domain error function after the low-pass filter as $\epsilon_{lp}(t) = f(t) - s_{lp}(t)$, that is the difference between the ideal output signal and the actually retrieved signal in the time domain. The minimization of the power of $\epsilon_{lp}(t)$ is chosen as the optimization criterion. Some long though straightforward calculations show that $\epsilon_{lp}(t)$ may be written as

$$\epsilon_{lp}(t) = \sum_{k=-K}^{K} e^{\mathbf{i}\, 2\pi f_s h t} \left[f_k - g_k \sum_{h=0}^{N-1} x_h e^{\mathbf{i}\, 2\pi k f_s h \Delta t} \right] \tag{15}$$

where $f_k = \pm \frac{A}{2\mathbf{i}}$ if $k = \pm 1$ are the Fourier series coefficients of (11), and zero for any other index k. Defining $\mathbf{x} = \{x_0, x_1, \ldots, x_{N-1}\}$ as the vector of N binary symbols x_i, the power of $\epsilon_{lp}(t)$ is a function of $\mathbf{x}$: $P(\mathbf{x}) = \frac{1}{T_s} \int_0^{T_s} \epsilon_{lp}^2(t) dt$. Carrying out the calculations and recalling the orthogonality of $e^{\mathbf{i}\, 2\pi f_s h_1 t}$ and $e^{\mathbf{i}\, 2\pi f_s h_2 t}$ for $h_1 \neq h_2$, one finds:

$$P(\mathbf{x}) = F + \mathbf{x}^T \underline{Q} \mathbf{x} - 2\mathbf{L}^T \mathbf{x} \tag{16}$$

where

$$F = \sum_{k=\pm 1} |f_k|^2 = \frac{A^2}{2}$$
$$q_{i,j} = \sum_{k=-K}^{K} |g_k|^2 e^{-\mathbf{i}\, 2\pi (i-j) \frac{k}{N}}$$
$$l_i = \text{Re} \left[\sum_{k=-K}^{K} g_k f_k^* e^{-\mathbf{i}\, 2\pi i \frac{k}{N}} \right].$$

In order to reduce high frequency switching losses, the number of swticthing inside the binary sequence pattern $\mathbf{x}$ has to be limited to a proper value n, exactly as it is in the traditional harmonic elimination methods. In formulas:

$$\frac{1}{2}\sum_{k=0}^{N-1}|x_{[k+1]_N} - x_k| = \frac{1}{4}\sum_{k=0}^{N-1}(1 - x_{[k+1]_N}\cdot x_k) = n \quad (17)$$

where $[\cdot]_N$ is the modulo-N operator, that keeps into account the periodicity of the output sequence.

Finally, since N represents the number of pulses inside a sinusoidal period T_s, and we are interested in solving the problem with $N \approx 500$ (that is an acceptable time resolution for motor drive application), we have to reduce the problem complexity to perform off-line calculations of the optimal sequence. For this purpose, we may rely on the fact that the optimal pattern $\mathbf{x}_{opt}$ has to represent a sine wave. Hence, owing to the sine wave quarter period symmetry, a quarter period symmetry for the sequence $\mathbf{x}$ follows as well, assuming that $N/4 \in \mathbf{N}$. In other words, the symmetry relation for the symbols x_i in (12) may be written as

$$x_i = \begin{cases} x_i & \text{if } 0 \le i \le N/4-1 \\ x_{N/2-1-i} & \text{if } N/4 \le i \le N/2-1 \\ -x_{i-N/2} & \text{if } N/2 \le i \le 3N/4-1 \\ -x_{N-1-i} & \text{if } 3N/4 \le i \le N-1 \end{cases} \quad (18)$$

So, the optimal switching sequence is the one that minimizes equation (16), subject to the constraints (17) and (18) for any fixed N and n. Note that the solution of such a minimization problem is an NP-Hard problem (in the dimension of $N/4$), i.e. it is intrinsically harder than those that can be solved by a Turing machine in polynomial time. In other words, the computational time to solve the problem grows exponentially with $N/4$. For this reason, we implemented an efficient backtracking branch-and-cut algorithm [7] that, making extensive use of FFT, finds $\mathbf{x}_{opt}$ for $N/4 \approx 250$.

4.1. Example

In this subsection we show an example of the previously described method to generate an optimal switching pattern, and we analyze the spectral properties of the retrieved signal at the low pass filter output. To make a fair comparison with the spectral properties of time quantized classical sequences, we set $n = 8$, $f_s = 50\text{Hz}$, and $A = 0.6\text{V}$. The time resolution is $\Delta t = 39\mu s$.The switching waveform is shown in figure 5 together with the retrieved one beyond an ideal low pass filter with cutoff frequency $B = 800\text{Hz}$. The PSD of the periodic waveforms is shown in 6. With this set of parameters, the harmonic distortion in the band B is $133\mu W$. So, this new technique exhibits an improvement of 6dBW in terms of spurious harmonic power, compared with the classical harmonic elimination techniques with time quantized switching instants.

5. Acknowledgments

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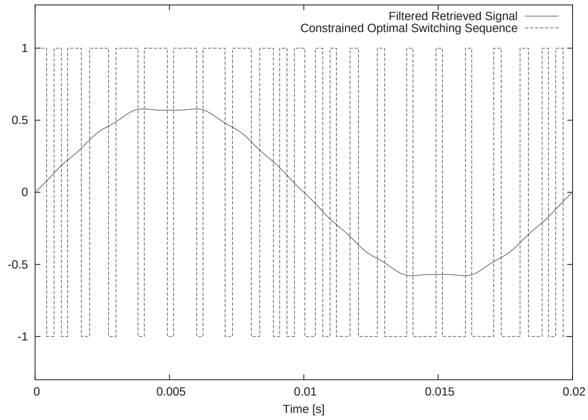

Figure 5: Optimal switching pattern and output waveform.

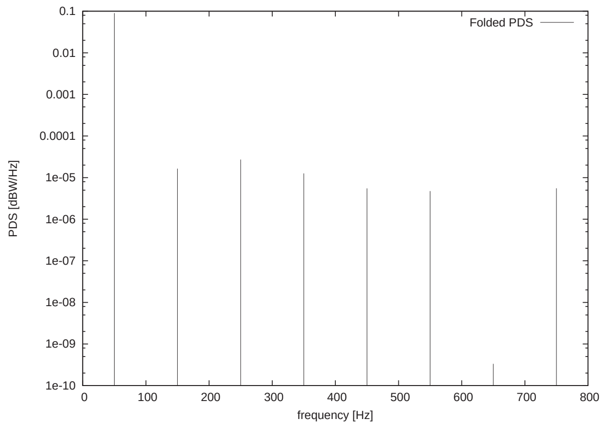

Figure 6: PSD of the optimal switching waveform.

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