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Analysis of Discretely Controlled Switched Positive Systems

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Abstract—Discretely controlled switched positive systems are characterized by interacting continuous and discrete dynamics. The continuous dynamics of the system ís positive. Vector field analysis is used to show that the trajectories of discretely controlled switched positive systems can be restricted to an invariant set away from the equilibrium points of the constituent systems making up the switched system. These invariant sets are called $\mathcal{H}$-invariant sets. Based on the properties of the $\mathcal{H}$-invariant sets, a design method to restrict the steady-state values of the continuous states to desired sets is given.

1. Introduction

This paper considers a new class of systems called discretely controlled switched positive systems that are characterized by interacting continuous and discrete subsystems. The change of the discrete state (switching) is effected by the controller. Switching must take place to move the continuous state from the initial state to a goal state. Furthermore, switching must continue even after the goal state has been reached in order to hold the state in the given goal region of the state space.

This class of systems is characterized by positive continuous dynamics. Examples of this type of systems include DC-DC converters [5] where switching must take place indefinitely in order to maintain the output voltage within a given range or processes where heated parts are used to transfer heat energy to other parts or fluids in order to maintain them within a given temperature range [4].

The difficulties of dealing with such system arise from the fact that these systems work in regions of the state space away from their equilibrium points. Due to this property, this class of systems is not stable in the Lyapunov sense. As a result, the large amount of literature dedicated to Lyapunov stability-based analysis of switched systems like [2] and [6] cannot be applied to these systems.

In this paper, phase plane analysis of the positive continuous dynamics is used to show the boundedness of the state of the discretely controlled switched positive system. Under the assumptions made the trajectories of the switched system cannot diverge to infinity regardless of the way the switching thresholds are selected. Furthermore, the continuous trajectories of the system cannot escape from the nonnegative part of the continuous state space. These concepts can be extended to higher dimensional systems.

Analysis of the steady-state behaviour of the trajectories shows that the trajectories of the discretely controlled switched positive systems can be restricted to invariant sets away from the equilibrium points of the constituent systems making up the switched system. These invariant sets are called $\mathcal{H}$-invariant sets. For planar systems, the trajectories within a $\mathcal{H}$-invariant set converge to a stable and unique limit cycle regardless of the initial state. This idea can be applied to design controllers which restrict the steady-state values of the continuous states to desired sets regardless of the initial state.

Due to space limitations, the propositions in this paper are stated without proofs; the reader can refer to [4] for the proofs.

2. The Model

2.1. Block diagram representation

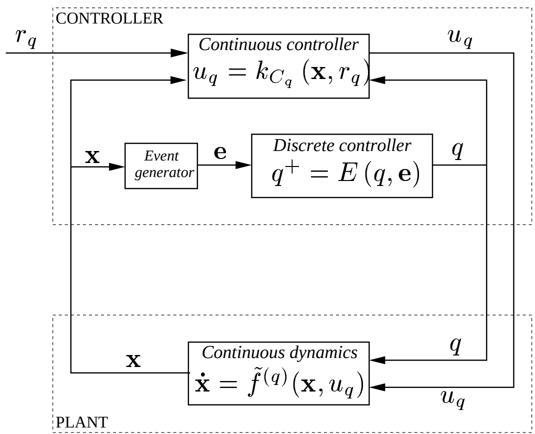

Figure 1: The model

Figure 1 shows the closed-loop system consisting of the plant and the controller. The symbols used in the figure are defined as follows:

- $\mathbf{x} = [x_1\ x_2\ \ldots x_n]^T \in X \subseteq R^n$ is the continuous state vector.

- $q \in Q \subseteq N^M$ is the discrete state prescribed by the discrete part of the controller.
- $u_q \in U \subseteq R$ is the continuous control signal vector.
- $\mathbf{e} = \{e_1, e_2, \ldots, e_M\}$, with $e_q \in \{0, 1\}$ is a vector of discrete events. The event e_q is generated when the continuous state $\mathbf{x}$ crosses the threshold $\Phi_q(\mathbf{x})$ in the continuous state space, i.e.

$$e_q = \begin{cases} 0 & \text{if} \quad \Phi_q(\mathbf{x}) \leq 0 \\ 1 & \text{otherwise} \end{cases} . \qquad (1)$$

- $r_q \in R$ is a reference input for the continuous controller.

There is a unique continuous map $\tilde{f}^{(q)}(\cdot)$ associated with each discrete state q, and the continuous dynamics which are active at any given time are determined by the discrete controller. There is a separate continuous controller for each discrete state, i.e. the continuous controller is switched together with the continuous plant.

2.2. Modelling assumptions

In this paper, the following modelling assumptions are made:

- The continuous state $\mathbf{x}$ does not change at the switching instants, i.e. there are no state jumps.
- Each of the continuous systems $\tilde{f}^{(q)}(\cdot), \quad q = 1, \ldots, M$ is a positive linear system of the form

$$\dot{\mathbf{x}} = \tilde{\mathbf{A}}^{(q)}\mathbf{x} + \tilde{\mathbf{b}}^{(q)}u_q. \qquad (2)$$

- The continuous controllers preserve the positivity of the associated continuous dynamics.
- Each of the continuous systems is observable.
- None of the continuous systems $\tilde{f}^{(q)}(\cdot), \quad q = 1, \ldots, M$ is completely state controllable by virtue of some inputs not being directly or indirectly connected to the states. This assumption is made to exclude the trivial solution where the switched system can be transferred from the initial state to the goal state without switching. It is assumed that the poles corresponding to the uncontrollable states lie strictly on the left-half plane.
- The switching surfaces $\Phi_q(\mathbf{x})$ are linear functions of the form $\mathbf{c}_q^T\mathbf{x} - d_q = 0, \ q = 1, \ldots, M$.

2.3. Specific continuous models considered

It is assumed that each of the positive continuous-time systems making up the switched system has only one controllable state, but the combination of all the controllable states for all the continuous systems covers all the n components of the continuous state space. In a 2-dimensional setting, the continuous systems are given by

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} \tilde{a}_{11}^{(1)} & \tilde{a}_{12}^{(1)} \\ 0 & \tilde{a}_{22}^{(1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \tilde{b}_1^{(1)} \\ 0 \end{bmatrix} u_1 \\ &= \tilde{\mathbf{A}}^{(1)}\mathbf{x} + \tilde{\mathbf{b}}^{(1)}u_1, \qquad (3) \\ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} \tilde{a}_{11}^{(2)} & 0 \\ \tilde{a}_{21}^{(2)} & \tilde{a}_{22}^{(2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \tilde{b}_2^{(2)} \end{bmatrix} u_2 \\ &= \tilde{\mathbf{A}}^{(2)}\mathbf{x} + \tilde{\mathbf{b}}^{(2)}u_2. \qquad (4) \end{aligned}$$

3. Analysis of the vector fields

Affine state feedback is used to stabilize the continuous systems while maintaining the positivity of the continuous dynamics, [4]. After the design of the continuous controller, the continuous dynamics of the asymptotically stable closed-loop systems corresponding to systems (3) and (4) are given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} \\ 0 & a_{22}^{(1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1^{(1)} \\ 0 \end{bmatrix} = \mathbf{A}^{(1)}\mathbf{x} + \mathbf{b}^{(1)}, \qquad (5)$$

and

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11}^{(2)} & 0 \\ a_{21}^{(2)} & a_{22}^{(2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b_2^{(2)} \end{bmatrix} = \mathbf{A}^{(2)}\mathbf{x} + \mathbf{b}^{(2)}. \qquad (6)$$

The continuous controllers are designed in such a way that

$$-a_{ii}^{(q)} > \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}^{(q)} \text{ for } i = 1, \ldots, n, \ q = 1, \ldots, n. \qquad (7)$$

Switching between systems (5) and (6) means superimposing the vector fields of the respective systems. If the continuous systems satisfy condition (7), the vector fields of the switched system are as shown in Fig. 2, [4]. Line l_1 is defined by the equation $a_{11}^{(1)} + a_{12}^{(1)} + b_1^{(1)} = 0$ while line l_2 is given by $a_{21}^{(2)} + a_{22}^{(2)} + b_2^{(2)} = 0$. The point $[\bar{x}_1, \ 0]^T$ is the equilibrium point of system (5) while point $[0, \ \bar{x}_2]^T$ is the equilibrium point of system (6). The arrows show the directions of the vector fields. The figure shows that the continuous state space $R_{\geq 0}^n$ can be partitioned into 4 different regions depending on the direction of the vector fields.

Proposition 3.1 *The non-negative part of the state-space (the set $R_{\geq 0}^n$) is an invariant set for the discretely controlled switched positive systems.*

Proposition 3.2 *The state of the discretely controlled switched positive system made up of the second-order asymptotically stable positive systems (5) and (6) is bounded, i.e.*

$$\|\mathbf{x}(0)\| < \infty \quad \Rightarrow \quad \|\mathbf{x}(t)\| < \infty \ \ \forall t \geq 0 \qquad (8)$$

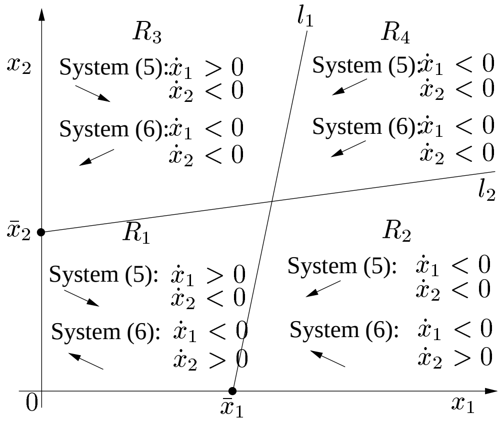

Figure 2: Vector fields of systems (5) and (6) superimposed

This concept of the boundedness of the state can be extended to higher dimensional systems provided the continuous controllers are designed such that each system satisfies condition (7), [4].

4. $\mathcal{H}$-invariant sets

It is well known for a linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{9}$$

PSfrag replacements

that a set $\tilde{X} \subseteq R^n$ is said to be $\mathbf{A}$-invariant if $\mathbf{A}\mathbf{x} \in \tilde{X}$ for all $\mathbf{x} \in \tilde{X}$. Every trajectory of system (9) starting from the $\mathbf{A}$-invariant set $\tilde{X}$ remains in that set for all future time, [1]. Analogously, a $\mathcal{H}$-invariant set ($\mathcal{H}$ for hybrid) is defined for discretely controlled switched positive systems:

Definition 4.1 *A set $H \subseteq R^n_{\geq 0}$ is called $\mathcal{H}$-invariant if all the trajectories of the discretely controlled switched positive system starting from H remain in that set for all future time.*

In the previous section, it was stated that the trajectories of the discretely controlled switched positive system cannot escape from the set $R^n_{\geq 0}$, which is the entire continuous state space for this type of system.

Corollary 4.1 *The set $R^n_{\geq 0}$ is the largest $\mathcal{H}$-invariant set for the discretely controlled switched positive system.*

To illustrate the properties of the $\mathcal{H}$-invariant set, consider Fig. 3. The switching thresholds Φ_1 and Φ_2 are straight lines originating along the x_1 axis in region R_1 and extending into region R_3. The switching surfaces do not intersect and do not enclose the equilibrium points $[\bar{x}_1,\ 0]^T$ or $[0,\ \bar{x}_2]^T$ between them. The setting in the figure assumes that the switched system has an initial state along the switching threshold Φ_2 with system (5) active. When the system trajectory crosses the switching threshold Φ_1, system (6) becomes active, and when the trajectory crosses threshold Φ_2, system (5) becomes active, and so on.

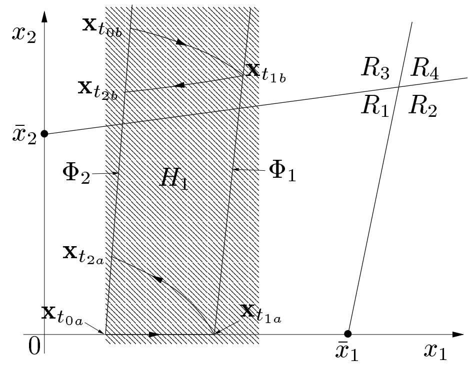

Figure 3: $\mathcal{H}$-invariant set H_1

The trajectory of the switched system starting at point $\mathbf{x}_{t_{0a}}$ along the x_1-axis in Fig. 3 evolves as shown in the figure, and after one cycle, the trajectory of the switched system ends up at a point $\mathbf{x}_{t_{2a}}$ which is higher than $\mathbf{x}_{t_{0a}}$. On the other hand, a trajectory of the switched system starting at point $\mathbf{x}_{t_{0b}}$ in region R_3 evolves as shown in Fig. 3, and after one cycle ends up at point $\mathbf{x}_{t_{2b}}$ which is lower than $\mathbf{x}_{t_{0b}}$ (this can be deduced from the directions of the vector fields shown in Fig. 2). It follows that trajectories of the switched system starting anywhere between the switching surfaces Φ_1 and Φ_2 (labelled as set H_1 in Fig. 3) cannot escape from that set, hence H_1 is a $\mathcal{H}$-invariant set.

If the switching thresholds Φ_1 and Φ_2 are chosen as explained above, it can be shown that there exists a special $\mathcal{H}$-invariant set L as shown in Fig. 4. The lower boundary of the set is the trajectory of system (5) from $\mathbf{x}_{t_0}$ to $\mathbf{x}_{t_1}$, the upper boundary is the trajectory of system (5) from $\mathbf{x}_{t_0^*}$ to $\mathbf{x}_{t_1^*}$, while the switching thresholds Φ_1 and Φ_2 form the side boundaries of the set.

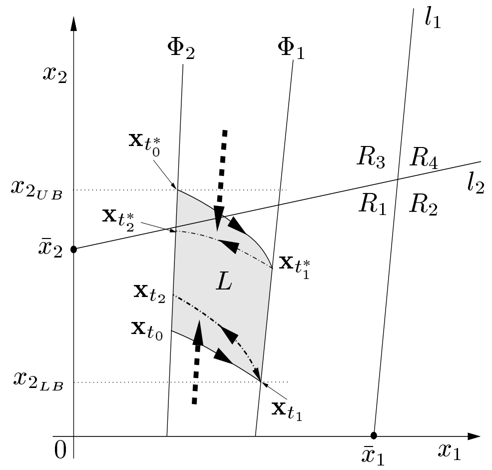

Figure 4: $\mathcal{H}$-invariant set L

The set L has the special property that with repeated switching, all trajectories starting below the lower boundary eventually enter the set from below and remain within the set for all future time. Similarly, all trajectories starting above the upper boundary eventually enter the set L from above and remain in the set for all future time. Trajectories starting within L remain within the set for all future time. Hence L is a $\mathcal{H}$-invariant set. Since the trajectories cannot escape from set L, the steady-state value of state x_2 is limited to an upper bound of $x_{2_{UB}}$ and a lower bound of $x_{2_{LB}}$ as shown by the dotted lines in Fig. 4.

Proposition 4.1 : *For a given set of switching surfaces Φ_1 and Φ_2, the $\mathcal{H}$-invariant set L shown in Fig. 4 contains one unique and stable limit cycle.*

The stability of the limit cycle can also be analyzed by use of a Poincaré map. The trajectory sensitivity matrix after exactly one period T of the limit cycle is known as the monodromy matrix, [3, 7], and the eigenvalues of this matrix determine the stability of the limit cycle. The monodromy matrix always has one eigenvalue of 1. If the other eigenvalues (known as characteristic multipliers or Floquet multipliers) are less than 1, the limit cycle is stable [7].

5. Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0.1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 100 \\ 0 \end{bmatrix} = \mathbf{A}^{(1)}\mathbf{x}+\mathbf{b}^{(1)},$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0.2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 270 \end{bmatrix} = \mathbf{A}^{(2)}\mathbf{x}+\mathbf{b}^{(2)}.$$

The objective is to maintain the steady-state value of state x_2 between $55 \leq x_2 \leq 60$. The switching surfaces therefore chosen as $\Phi_1'(\mathbf{x}) = -x_2 + 55 = 0$ and $\Phi_2'(\mathbf{x}) = x_2 - 60 = 0$. For this set of switching surfaces, the vertices of the limit cycle are computed as $\mathbf{x}_0 = [34.5150,\ 55]^T$ and $\mathbf{x}_1 = [31.3477,\ 60]^T$. The monodromy matrix is found to be

$$\varphi(\mathbf{x}, T) = \begin{bmatrix} 0.86877 & -0.08095 \\ 0.00992 & 1.00612] \end{bmatrix}$$

with eigenvalues of 1 and 0.87489 hence the limit cycle is stable.

A new set of switching surfaces $\Phi_1(\mathbf{x})$ and $\Phi_2(\mathbf{x})$ is selected as $\Phi_1(\mathbf{x}) = x_1 - 34.5150 = 0$ and $\Phi_2(\mathbf{x}) = -x_1 + 31.3477 = 0$. These switching surfaces pass through the vertices $\mathbf{x}_0$ and $\mathbf{x}_1$ of the limit cycle computed above. The vertices of the limit cycle are found to be $\mathbf{x}_0 = [34.5150,\ 55]^T$ and $\mathbf{x}_1 = [31.3477,\ 60]^T$, which are the same as before. For this set of switching surfaces, the monodromy matrix is

$$\varphi(\mathbf{x}, T) = \begin{bmatrix} 0.99444 & -0.00343 \\ -0.31699 & 0.80446 \end{bmatrix}$$

with eigenvalues of 1 and 0.7989 hence this limit cycle is also stable.

The trajectory of state x_2 plotted against time is shown in Fig. 5, and it can be seen that the steady-state value of state x_2 is restricted to the desired range of $55 \leq x_2 \leq 60$. Furthermore, the switched system must switch indefinitely to maintain the value of state x_2 within this range.

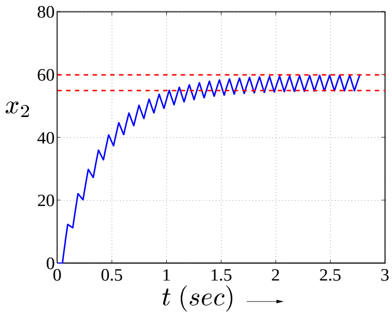

Figure 5: State x_2 against time

6. Conclusion

It was shown that the trajectories of the discretely controlled switched positive system can be restricted to invariant sets called $\mathcal{H}$-invariant sets. For a planar system, the trajectories within an $\mathcal{H}$-invariant set converge to a stable and unique limit cycle as long as the initial state is chosen between the switching surfaces.

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