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Symbolic Analysis: A Practical Alternative Viewpoint of Border Collision

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Abstract— In this paper, a symbolic method is proposed for analyzing the bifurcation in switching power converters. This method focuses on the cyclic operation of the system which characterizes its bifurcation behavior. The concept of *block sequence* is introduced, which, in conjunction with the periodicity of the system, can be used to distinguish the various types of bifurcation behavior, e.g., "smooth" period doubling and "nonsmooth" border collision. The proposed method is applied, as an illustrative example, to develop 2-dimensional bifurcation diagrams for a voltage-mode controlled buck converter.

1. Introduction

In much of the previous study of nonlinear phenomena in switching power converters, a variety of complex behavior such as bifurcation and chaos has been identified [1, 2, 3]. Typically, switching power converters undergo topological changes cyclically in time. Although each involving circuit topology is linear, the overall dynamics of a switching power converter can be rather complex.

Traditional methods of analyzing bifurcation involve a typical stability evaluation, as most "standard" bifurcations are characterized by a change of stability status [4]. Methods pertaining to stability tests (e.g., inspecting eigenvalues of Jacobians) are therefore effective only for studying bifurcations that involve a loss of stability status. For border collision, which is also typical of switching converters, such methods of analysis are either irrelevant or inapplicable. In fact, border collision occurs as a consequence of a "structural change" as one or more parameters are changed. In the case of switching converters, this translates to a "change of topological sequence" as one or more parameters are varied [2]. In this paper, we propose a symbolic analytical approach, whereby the dynamics of the topological sequence is examined. We introduce a new concept of *block sequence* which can be used to distinguish border collision from other standard bifurcations.

2. Review of Bifurcation in Switching Power Converters

Generally speaking, there are two different types of bifurcation exhibited by switching power converters:

- *Standard bifurcation,* which is characterized by a change of stability status, e.g., period-doubling, saddle-node and Hopf bifurcations.
- *Border collision,* which is characterized by a change of operation as a result of a change of topological sequence.

Standard bifurcations are also known as "smooth" bifurcations because they arise from smooth dynamical systems which do not experience any structural change at the bifurcation point. Border collisions, moreover, are regarded as "nonsmooth" bifurcations because the functions used to describe their dynamics are nonsmooth at the bifurcation point. In switching converters, we may attribute any border collision to a change of topological sequence.

Unlike standard bifurcations, border collision does not have a universal manifestation.[1] It can be a transition from period-1 operation to period-n operation, or from period-n operation to chaos, etc. [5]. The differences between these two types of bifurcations can also be viewed from their bifurcation diagrams. Border collision usually manifests itself as some abrupt transitions, e.g., abrupt bendings, discontinuities, and jumps.

In the following, we will introduce a symbolic analysis method which makes use of the fundamental cause of border collision in terms of topological changes to distinguish various types of bifurcations in switching power converters.

3. Symbolic Method for Bifurcation Analysis

In switching power converters, the switch and the diode act as switching elements. If the number of switching elements is N, there will be 2^N possible switching states. In practice, however, not all switching states are used. The switching states that are relevant to a particular operation depend on the control scheme applied and the conduction mode of the system. If we inspect the system in a particular period T, where T is the period of the driving clock, we see that the circuit takes a sequence of circuit topologies. It is clear that the topological sequence in a switching period governs the dynamics of the system.

For clarity of discussion, the following definitions apply to all switching converters.

[1] Standard bifurcations usually have characteristic universal manifestations. For example, period doubling manifests as a doubling of the repetition period, and Hopf bifurcation manifests as sudden death of a fixed point and birth of a limit cycle or quasi-periodic orbit.

Definition 1 *A* **switching block** *is a sequence of switch states which is taken within one particular switching period.*

Definition 2 *A* **block sequence** *is a symbolic sequence of switching blocks that describes the way in which the block of switch states changes as time advances.*

When studying the nonlinear phenomena of switching converters, discrete-time iterative maps are often used to describe the dynamics of the system [6, 7]. If the sampling is uniform to the driving clock, the iterative map is called a stroboscopic map, which is widely used in much of the previous work. The block sequence defined above can be viewed as being derived from a generalized stroboscopic map. It deals with the switching states of the system, but not the values of the state variables.

By this definition, a periodic or aperiodic solution can be transformed into an infinite sequence of switching blocks. Obviously, for any periodic solution, its block sequence must be periodic. Moreover, for an aperiodic solution, its block sequence may be aperiodic or periodic. Hence, a *periodic block sequence does not imply a periodic solution, but an aperiodic block sequence will imply an aperiodic solution.*

For instance, if b is a specific switching block, then bbb... is a periodic block sequence, but it does not necessarily imply a periodic solution of the system.[2] To simplify the description of various block sequences, we use the following notations.

Definition 3 *Let b_1, b_2, ..., b_m be switching blocks. We denote by $(b_1 b_2 \cdots b_m)_n$ a finite block sequence which repeats the block sequence $(b_1 b_2 \cdots b_m)$ n times. Moreover, a periodic block sequence is denoted as $(b_1\ b_2 \cdots b_m)_\infty$, and an aperiodic block sequence as (∞).*

3.1. Detecting and Distinguishing Border Collision and Standard Bifurcations

As mentioned in the previous section, border collision is caused by a structural change of the system which is equivalent to a change in the topological sequence. Hence, the block sequence of the system must experience a qualitative change when the system undergoes a border collision. This basic fact is summarized by the following theorem which is useful for detecting border collision.

Theorem 1 *Consider a switching power converter with parameter $\alpha \in \boldsymbol{R}$. Suppose the block sequence for $\alpha < \alpha_c$ is $\boldsymbol{B}_1$ and the block sequence for $\alpha > \alpha_c$ is $\boldsymbol{B}_2$.[3] Then, border collision occurs at $\alpha = \alpha_c$ if $\boldsymbol{B}_1 \neq \boldsymbol{B}_2$.*

[2] As we will see, this definition is useful in distinguishing border collision from standard bifurcations.

[3] The condition $\alpha < \alpha_c$ is a local condition and should be more rigorously written as $\alpha = \alpha_c - \epsilon$ for all $0 < \epsilon < \epsilon_0$ for some positive ϵ_0. The condition $\alpha > \alpha_c$ can be likewise understood. However, we write $\alpha < \alpha_c$ and $\alpha > \alpha_c$ in the theorem statement for better readability.

The above theorem follows directly from the mechanism of border collision. Specifically, border collision is characterized by the sudden loss of an operation and a simultaneous acquisition of a new operation. This is equivalent to a change in the system structure which alters the describing function of the system, hence border collision occurs. In switching converters, any operational change pertaining to a change of structure must be due to a change in the topological sequence. Hence, a change of block sequence implies border collision.

By inspecting the block sequence, the occurrence of border collision can be easily detected. In particular, we have the following observations in applying block sequence analysis.

1. Since the block sequence only provides partial information about the dynamics of the system, the exact manifestation of a border collision is not available.

2. Standard bifurcation (e.g., period doubling and Hopf) cannot be identified because the block sequence does not change when a standard bifurcation occurs.

To cover standard bifurcations in our analysis, extra information is required. We note that for periodically driven (nonautonomous) systems, the periodicity is a natural attribute of the system dynamics. Information about the periodicity can be easily obtained by sampling the waveforms.

We denote by P_w the periodicity of the system (precisely the periodicity of the waveforms). For instance, $P_w = n$ for a period-n operation.

Theorem 2 *Consider a switching converter with parameter $\alpha \in \boldsymbol{R}$. Suppose P_{w1} and $\boldsymbol{B}_1$ are, respectively, the periodicity and block sequence of the converter for $\alpha < \alpha_c$, and P_{w2} and $\boldsymbol{B}_2$ are, respectively, the periodicity and block sequence of the converter for $\alpha > \alpha_c$. Then, a standard bifurcation occurs at $\alpha = \alpha_c$ if $P_{w1} \neq P_{w2}$ and $\boldsymbol{B}_1 = \boldsymbol{B}_2$.*

This theorem can be reasoned as follows. First, from Theorem 1, if the block sequence remains unchanged, no border collision occurs. Moreover, the change of P_w implies a bifurcation, and this bifurcation must be a standard bifurcation.

Remarks on Periodicity — The term periodicity may have two different interpretations. The first one is *waveform periodicity* P_w, which was discussed earlier. The other interpretation is the *block sequence periodicity,* denoted by P_s. For a periodic solution with $P_w = n$, the block sequence periodicity P_s is a common divisor of n. Thus, $P_s \leq n$.

We note that waveform periodicity implies block sequence periodicity, but not vice versa. Thus, even if the block sequence is periodic, the waveforms (system) can be aperiodic.

3.2. Analysis Procedure

Based on Theorems 1 and 2, a procedure for symbolic analysis can be derived. Basically, we observe the block se-

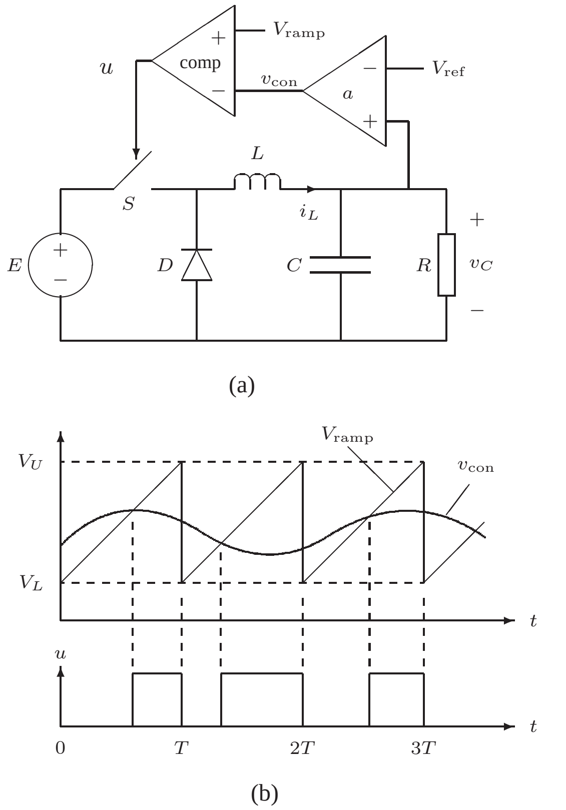

Figure 1: Voltage-mode controlled buck converter. (a) Circuit; (b) typical waveforms illustrating the operation.

quence and waveform periodicity, and detect their variation as a selected parameter α is varied.

Suppose $\alpha_1 < \alpha_c < \alpha_2$, where α_c is the critical value. Denote the block sequence and the waveform periodicity, respectively, by $\boldsymbol{B}_i$ and P_{wi} for $\alpha = \alpha_i$ with $i = 1$ or 2. Then, we apply the above theorems to determine the type of bifurcation.

- If $\boldsymbol{B}_1 = \boldsymbol{B}_2$ (the block sequence is unchanged) and $P_{w1} \neq P_{w2}$, a standard bifurcation occurs at $\alpha = \alpha_c$. For example, period doubling occurs if $2P_{w1} = P_{w2}$. An expansion in the periodicity (i.e., $P_{w1} \ll P_{w2}$) may indicate a Hopf bifurcation.

- If $\boldsymbol{B}_1 \neq \boldsymbol{B}_2$ (the block sequence changes), border collision occurs at $\alpha = \alpha_c$. The information about P_{w1} and P_{w2} determines the manifestation of the border collision. For example, if P_{w1} is finite and equal to P_{w2}, border collision takes place to bring a periodic orbit to another periodic orbit of the same period. Also, if $P_{w1} \neq P_{w2}$, and P_{w1} and P_{w2} are finite, border collision transmutates a periodic orbit to another periodic orbit of a different period. Moreover, if $P_{w1} \neq P_{w2}$ with P_{w1} being finite and P_{w2} infinite, border collision occurs to transmutate a periodic orbit to a chaotic one.

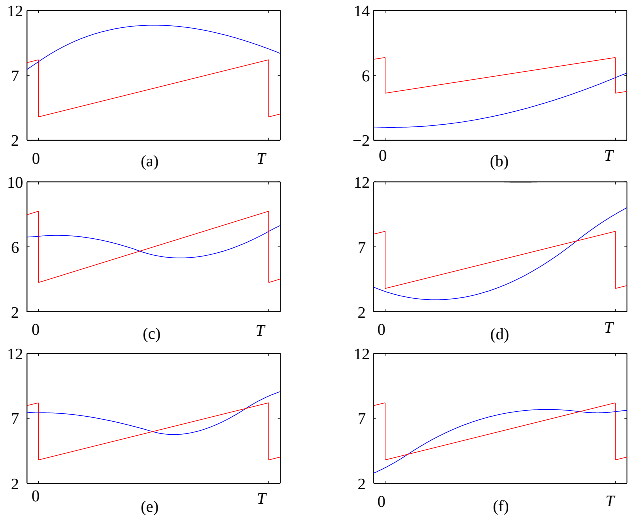

Figure 2: Illustrative waveforms for different types of blocks. (a) Block **1** ($i = 0$); (b) block **2** ($i = 0$); (c) block **3** ($i = 1$); (d) block **4** ($i = 1$); (e) block **5** ($i = 2$); (f) block **6** ($i = 2$).

4. Application Example

In this section, the symbolic analysis procedure is applied to the voltage-mode controlled buck converter shown in Fig. 1 (a). The operation of buck converter under study can be briefly described as follows [2, 3, 8]. The output voltage error with respect to the reference voltage is amplified to give a control voltage $v_{\rm con}$ as

$$v_{\rm con}(t) = a(v_C(t) - V_{\rm ref}), \qquad (1)$$

where a is the feedback amplifier gain and $V_{\rm ref}$ is the reference voltage. Then, switch S is controlled by comparing the control voltage $v_{\rm con}$ with a ramp signal $V_{\rm ramp}$. The ramp signal is given by

$$V_{\rm ramp}(t) = V_L + (V_U - V_L)\left(\frac{t}{T} \bmod 1\right), \qquad (2)$$

where V_L and V_U are the lower and upper voltages of the ramp, respectively. The comparator output, u, gives the pulse-width-modulated signal necessary for driving the switch. Typically, switch S is turned on when $v_{\rm con}(t) \leq V_{\rm ramp}$, and turned off when $v_{\rm con}(t) > V_{\rm ramp}$, as illustrated in Fig. 1 (b). For simplicity, we consider continuous conduction mode (CCM), in which the inductor current never drops to zero. To guarantee operation in CCM, the parameters are chosen as follows:

$$L = 20 \text{ mH}, C = 47\ \mu\text{F}, T = 400\ \mu\text{s}, a = 8.4,$$
$$V_{\rm ref} = 11.3 \text{ V}, V_L = 3.8 \text{ V and } V_U = 8.2 \text{ V}$$

Furthermore, the load resistor R and the input voltage E are varied simultaneously and taken as bifurcation parameters.

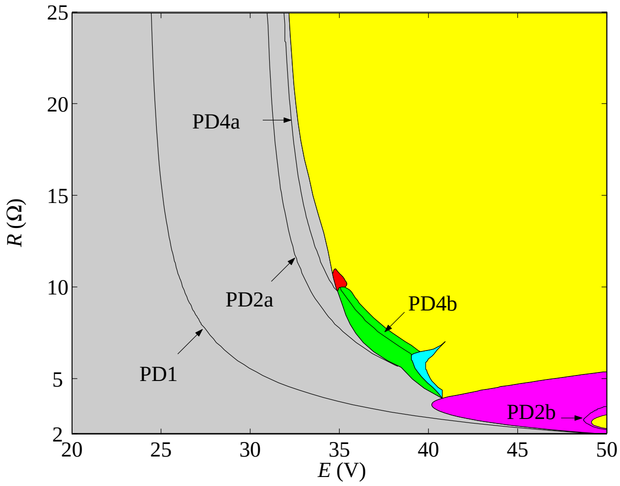

Figure 3: Bifurcation diagram for the voltage-mode controlled buck converter in the parameter plane $\{(R, E) : 2 \leq R \leq 25;\ 20 \leq E \leq 50\}$ showing regions of operation with various switching block sequences. Legends: grey $= (\mathbf{3})_\infty$; magenta $= (\mathbf{35})_\infty$; red $= (\mathbf{13333333})_\infty$; green $= (\mathbf{1333})_\infty$; cyan $= (\mathbf{13331335})_\infty$; yellow = the rest including chaotic and periodic regions. Boundary curves separating regions of different colors locate the occurrence of border collision. Within some specific region, some bifurcation boundary curves corresponding to PD1, PD2a, PD2b, PD4a and PD4b are also plotted. These curves locate the standard period doubling occurs.

According to the circuit operation, no switching or any number of switchings in a switching period are allowed. If there are i switching actions in a switching period T, the corresponding switching block will be a sequence of $i+1$ switch states. Thus, the block can be conveniently defined as

$$\text{block} = \begin{cases} 2i+1 & \text{if the first switch state is OFF.} \\ 2i+2 & \text{if the first switch state is ON.} \end{cases} \tag{3}$$

Fig. 2 shows the waveforms corresponding to some possible switching blocks.

Using our symbolic method, bifurcation diagrams can be obtained, as exemplified in Fig. 3. For simplicity, only some typical regions are shown in different colors in the diagram. From Theorem 1, it can be easily deduced that border collision occurs on the border which separates different regions. In addition, some typical bifurcation boundaries, across which the doubling of periodicity P_w occurs from left to right, are also given. These curves are denoted as PDni, where n is the shorter P_w associated with the period doubling, and i is an index to distinguish bifurcations with the same n. Since every curve is located inside a specific region (for instance, PD1, PD2a and PD4a in the grey region, PD2a in the magenta region, PD4a in the green region), standard period doubling, from Theorem 2, will occur when the parameters move across the curve. Moreover, the exact type of border collision can also be obtained if the information about the periodicity P_w on both sides of the curve is available. For example, $P_w = 4$ in both sides of the upper boundary curve between the grey region and the green region. Thus, border collision takes place with block sequence $(\mathbf{3})_\infty$ being transmuted to $(\mathbf{1333})_\infty$ with the period unchanged. Moreover, for the lower boundary curve between the grey region and the green region, $P_w = 2$ in the left side and $P_w = 4$ in the right side. Hence, border collision occurs with block sequence $(\mathbf{3})_\infty$ being transmuted to $(\mathbf{1333})_\infty$, together with a period-doubling manifestation.[4]

5. Conclusion

We have introduced in this paper a method for analyzing the bifurcation behavior of switching power converters. Unlike the conventional methods, our method focuses on the operational change when the system undergoes a specific bifurcation. The concept of block sequence is used to describe the operational condition of the system. Combining with the periodicity information, both standard bifurcation and border collision can be identified.

Acknowledgment

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[4] For a more detailed exposition of the various bifurcation possibilities in the voltage-controlled buck converter, refer to [8].