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Bifurcation of the Chaotic Attractor in a Simple Piecewise Smooth Systems

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Abstract—This paper studies the bifurcation phenomena of the chaotic attractor in a simple continuous system corresponding to the piecewise smooth system. We show the circuit model and derive a return map. Using a return map, the chaotic attractor can be analyzed in a parameter plane rigorously. Some theoretical results are verified by the laboratory experiments.

1. Introduction

The piecewise smooth system shows several interesting phenomena, both numerically and experimentally. In particular, intensive study on switched dynamical system has been devoted to the analysis of the dynamics in the continuous systems corresponding to the piecewise smooth system. In the simplest piecewise smooth system, the map has a differential function form for two region, but it is continuous across the border and its derivative is discontinuous. This system exhibits many interesting bifurcation phenomena and various interesting results have been published: analysis of 1 and 2-dimensional piecewise smooth maps[1][2], another approach for this problem called C-bifurcations[3][4], theoretical and experimental study of power electronic systems[5], and so on[7]. On the other hand, the power electronic system is the typical class of the piecewise smooth system. Over the past years, a great deal of studies have been devoted to one and two dimensional continuous system corresponding to the piecewise smooth system which consists of two regions[8][9]. Since current or voltage controlled converters have wide industrial applications, the analysis of the bifurcation phenomena for the piecewise smooth system is a basic problem in practical viewpoint. However, why does m-piece chaotic attractor change other type chaotic attractor in the continuous system corresponding to the piecewise smooth system?

The purpose of this paper is to study the bifurcation phenomena of the chaotic attractor in a simple continuous system corresponding to the piecewise smooth system. We show the circuit model and derive a return map. Using a return map, the chaotic attractor can be analyzed in a parameter plane rigorously.

2. Simple Piecewise Smooth System

First, we briefly explain the behavior of the circuit and its dynamics. Figure 1(a) shows a simple circuit controlled by a switch. By rescaling $\tau = 1/(RCt)$, the system dynamics is described by

$$\frac{dv}{d\tau} = \begin{cases} -v + E, & \text{(SW: A)}, \\ -v, & \text{(SW: B)}. \end{cases} \tag{1}$$

where we relabel the clock pulse $T' = 1/(RC)T$ as T. In our system, we assume that the supplied DC voltage source is interchanged by changing positions of the switch and only state variable is the capacitance voltage v. When the v reaches the reference value v_{r}, the switch is turned toward B. Any clock pulses during this time is ignored. The switch keeps B until arrival of the next clock pulse(See Fig. 1(b)). We assume that the initial value v_k at a time kT and the switch is turned toward A. As a result, when the initial value is $v = v_k$ at the time kT, v_{k+1} at the time $(k+1)T$ is

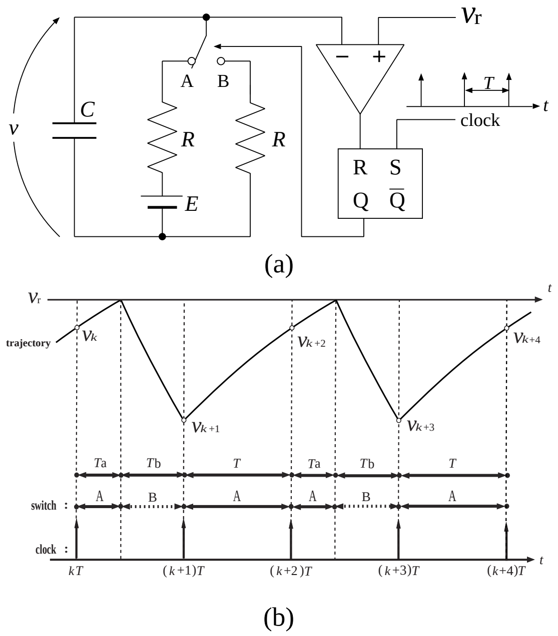
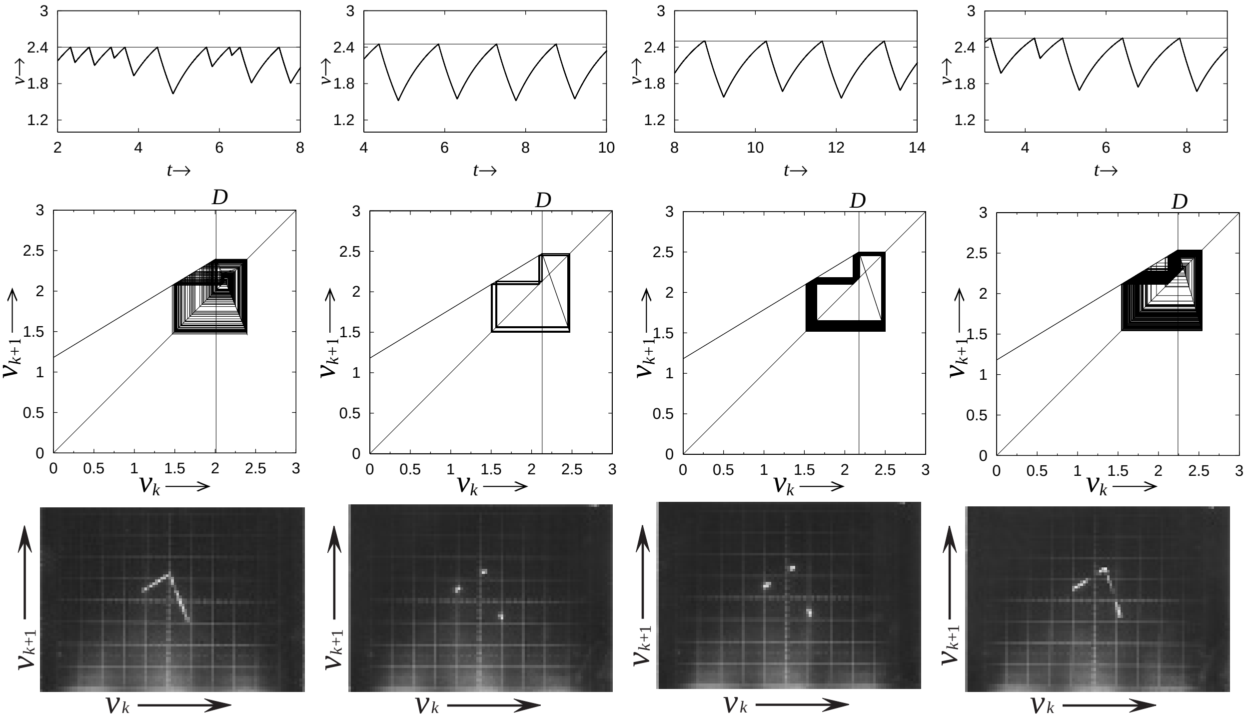


Figure 1: Circuit model and its behavior.



(a) m=1(v_{r}=2.4) (b) m=6(v_{r}=2.45) (c) m=3(v_{r}=2.5) (d) m=1(v_{r}=2.55)

Figure 2: Numerical and experimental results of m-piece chaotic attractor(T=0.485).

given by:

$$F(v_k) = v_{k+1} = \begin{cases} (v_k - E)e^{-T} + E, & v_k \le D, \\ v_{\mathrm{r}} \dfrac{v_k - E}{v_{\mathrm{r}} - E} e^{-T}, & v_k > D, \end{cases} \tag{2}$$

where $D = (v_{\mathrm{r}} - E)e^{T} + E$. Consequently, Eq. (1) can be interpreted as behavior of a discrete map. The derivative of the return map $F(v_k)$ is given by

$$DF(v_k) = \begin{cases} e^{-T} & = a, \quad v_k \le D, \\ \dfrac{v_{\mathrm{r}} e^{-T}}{v_{\mathrm{r}} - E} & = b, \quad v_k > D. \end{cases} \tag{3}$$

Let $I \equiv [F^2(D), F(D)]$. If $F^2(D) < F^3(D)$, $F(I) \subseteq I$. This means if $(a + 1)(1 - b) < 1$ is filled, we refer to I as an invariant interval of F. By varying the parameters, we can find many sub-harmonic bifurcation sets including border collision bifurcation.

3. Bifurcation Analysis

This system has a chaotic attractor with 1-, m-, or 2m-pieces. Figure 2 shows numerical and experimental results of the chaotic attractor. When the bifurcation phenomena from the chaotic attractor to another chaotic attractor occurs, these phenomena can be divided into three classes. Figure 3 shows the bifurcation structure of the chaotic attractor.

case A After 1-piece chaotic attractor there is a possibility(depending on the parameters) that, m-periodic orbit, m-piece chaotic attractor, or 2m-piece chaotic attractor can appear, or vice versa.

case B From 2m-piece chaotic attractor to m-piece chaotic attractor, or vice versa.

case C From m-piece chaotic attractor to 1-piece chaotic attractor, or vice versa.

Figure 4 shows two-parameter bifurcation diagram inside the non-periodic orbit. In a bifurcation diagram, m-periodic points and m-piece chaotic attractor are indicated by symbols mP and mC, respectively. In the following, we analyze the bifurcation mechanism of the chaotic attractor from a strictly mathematical viewpoint. Note that Fig. 4 is calculated mathematically by using the exact solution.

3.1. case A

When the following relation is satisfied, we can observe a border collision bifurcation of 1-piece chaotic attractor.

$$f^m(D) = D. \tag{4}$$

As a result, 1-piece chaotic attractor suddenly disappeared. After that, there is a possibility(depending on the parameters) that, m-periodic orbit, 2m-piece chaotic attractor, or m-piece chaotic attractor can appear, or vice versa(See case

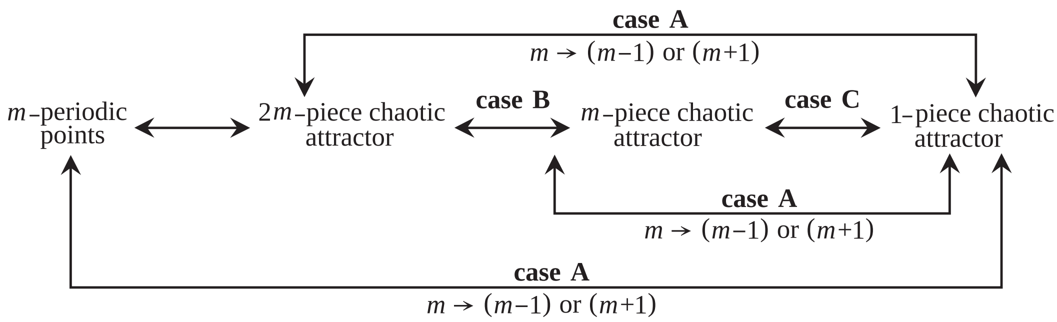


Figure 3: The bifurcation structure of the chaotic attractor.

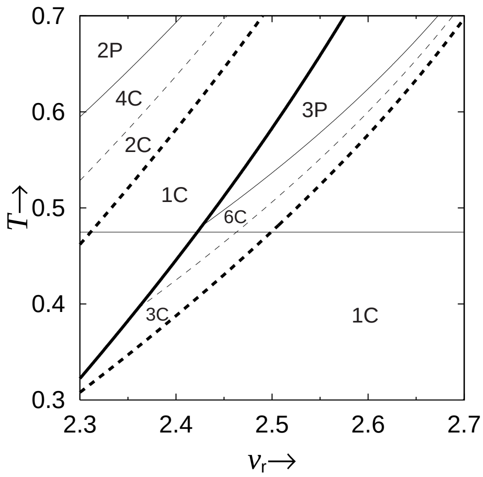


Figure 4: Two-parameter bifurcation diagram.

A in Fig. 3). In Fig. 4 with $T = 0.485$, when 1-piece chaotic attractor corresponding Fig. 2 (a) crosses the condition (4), 1-piece attractor suddenly changes a strange attractor corresponding Fig. 2 (b). After the long transient response, the orbit settle into 6-piece chaotic orbit which looks like 6-periodic points. We can also observe the others bifurcations (See. a thick solid line in Fig. 4).

3.2. case B

In Fig. 2 (b) and (c), we can observe a crisis from $2m$-piece chaotic attractor to m-piece chaotic attractor with m=3, or vice versa(See case B in Fig. 3). Figure 5(a) corresponding to Fig. 2(b) shows the return maps of v_k-v_{k+6}. Now, we focus on a part of the return map. When the following relationship is satisfied, $2m$-piece invariant intervals disappear via crisis.

$$f^{4m}(D_1) = F^{6m}(D_1), \tag{5}$$

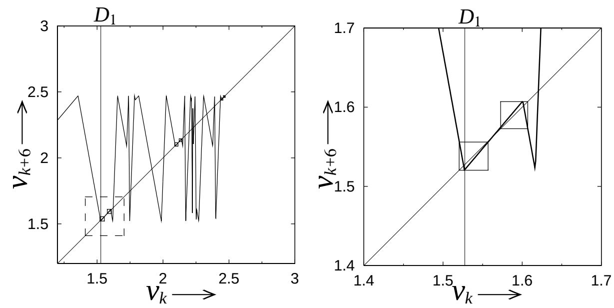
where

$$D_1 = \frac{(v_\mathrm{r} - E)^2}{v_\mathrm{r}} e^{(2m-1)T} + E\frac{v_\mathrm{r} - E}{v_\mathrm{r}} e^{mT} + E. \tag{6}$$

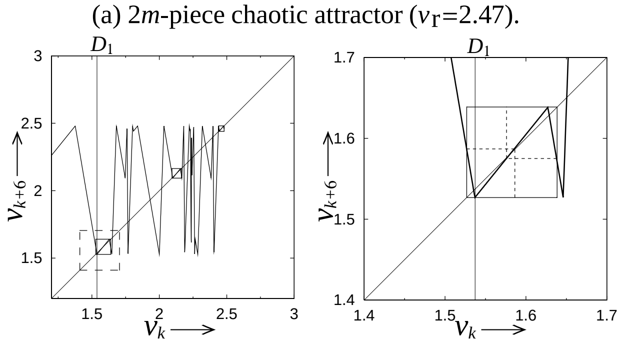
The periphery region of the border D_1 is described by

$$v_{k+2m} = F^{2m}(v_k) = \begin{cases} (v_k - E)\left(\frac{v_\mathrm{r}}{v_\mathrm{r}-E}\right)^3 e^{-2mT} \\ \quad -E\left(\frac{v_\mathrm{r}}{v_\mathrm{r}-E}\right)^2 e^{-mT} - \frac{v_\mathrm{r}E}{v_\mathrm{r}-E} e^{-T}, v_k \leq D_1, \\ (v_k - E)\left(\frac{v_\mathrm{r}}{v_\mathrm{r}-E}\right)^2 e^{-2mT} \\ \quad -\frac{v_\mathrm{r}E}{v_\mathrm{r}-E} e^{-mT}, v_k > D_1. \end{cases}$$

In Fig. 4 with $T = 0.485$, when the reference value v_r increases from this parameter, we can observer the bifurcation phenomena from 6-piece chaotic attractor to 3-piece chaotic attractor. Figure 5(b) shows the return maps of 3-piece chaotic attractor. Thin dashed line in Figure 4 indicates the condition (5).



(a) $2m$-piece chaotic attractor (v_r=2.47).



(b) m-piece chaotic attractor (v_r=2.48).

Figure 5: Return map of v_k-$F^{2m}(v_k)$ (T=0.485, m=3).

3.3. case C

In Fig. 2 (c) and (d), we can observe a crisis from m-piece chaotic attractor to 1-piece chaotic attractor with m=3, or vice versa. Figure 6(a) corresponding to Fig. 2(c) shows the return maps of v_k-v_{k+3}. We focus on a part of the return map. When the following relationship is satisfied, m-piece invariant intervals disappear via crisis.

$$F^{2m}(D_2) = F^{3m}(D_2), \tag{7}$$

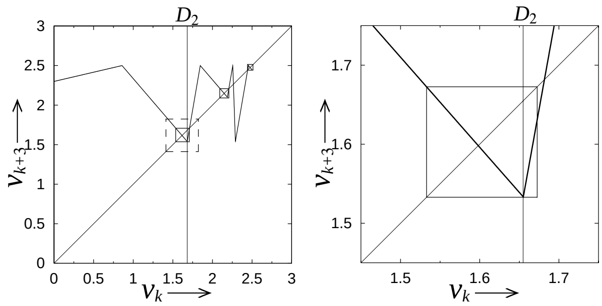
where

$$D_2 = (v_{\mathrm{r}} - E)e^{(m-1)T} + E. \tag{8}$$

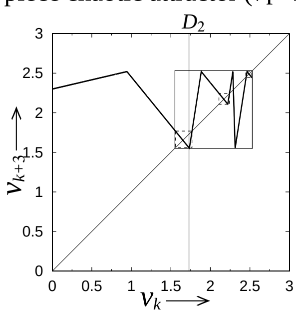
The periphery region of the border D_2 is given by

$$v_{k+m} = F^m(v_k) = \begin{cases} v_k e^{-mT}, v_k \le D_2, \\ (v_k - E)\left(\frac{v_{\mathrm{r}}}{v_{\mathrm{r}}-E}\right)^2 e^{-mT} - \frac{v_{\mathrm{r}}E}{v_{\mathrm{r}}-E}e^{-T}, v_k > D_2. \end{cases}$$

When the reference value v_{r} increases from Fig. 2 (c), 3-piece chaotic attractor satisfies $F^6(D_2) = F^9(D_2)$. As a result, 3-piece invariant intervals disappear via crisis. After that 1-piece chaotic attractor generates(See 6(b)). Thick dashed line in Fig. 4 indicates the condition (7).



(a) 3-piece chaotic attractor (v_{r}=2.5).



(b) 1-piece chaotic attractor (v_{r}=2.48).

Figure 6: Return map of v_k-$F^6(v_k)$ (T=0.52).

4. Conclusions

We have studied the dynamics of simple continuous system corresponding to the piecewise smooth system. By varying the amplitude of an input voltage and the period of the clock pulse, we found many bifurcation sets in parameter plane. Using a return map, the bifurcation mechanism of the chaotic attractor was analyzed. Some theoretical results were confirmed by laboratory measurements.

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