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General Consideration for Modelling and Analyzing Switched Dynamical Systems

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Abstract—Previously, switched dynamical systems have been studied in terms of switching that occurs at a common border between two regions in the same space as the system trajectory crosses the border. However, models arising from this consideration cannot cover systems whose trajectories do not actually “cross” the border. A typical example is the current-mode controlled boost converter whose trajectory is “reflected” at the border. In this paper, we propose a general method to model switched dynamical systems. Also, we suggest an analytical procedure to determine periodic solutions and their stability. The method is developed in terms of solution flows and no solution has to be explicitly written. Most practical switched dynamical systems can be covered by this modelling and analytical method.

1. Introduction

Problems associated with switched dynamical systems are important not only in theoretical context but also in practice. Many practical systems, such as power converters, chaos generators, variable structure controllers, etc., can be modelled as switched dynamical systems. In recent decades, research into this topic and its applications has attracted much attention.

Up to now, models for studying switched dynamical systems have focused on “switchings” as the system trajectory crosses some borders in the state space. Essentially, one or more pre-defined common borders divide the state space into two or more separate regions. The dynamics in different regions are governed by different system equations. As the trajectory crosses any border and moves into another region, switching occurs and the system is redefined [1–3], as illustrated in Fig. 1.

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Several questions arise from this modelling approach. Does the trajectory move in the same space all the time? Does it always move across the border as it hits the border?

An example of the current-mode controlled boost converter may shed some light on the modelling problem. Referring to Fig. 2(a), the switch is turned off when the inductor current i_L is equal to a reference current I_{ref}. A clock signal turns on the switch periodically at $t = kT$. During the on-time, i_L climbs to the value of I_{ref}, and then ramps down during the off-time. A typical waveform of the inductor current is shown in Fig. 2(b). Thus, we can see that (i) the solution never crosses the “border” at the turn-off instant, instead it is being “reflected” there; (ii) one border is available for one state, i.e., border is not common. To cover such systems, we need to revise the modelling method.

To analyze the dynamical behavior of a system, periodic solutions and their stability are often consid-

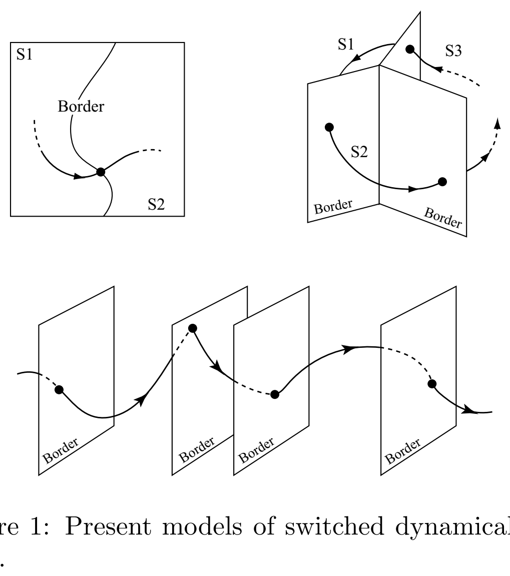

Figure 1: Present models of switched dynamical systems.

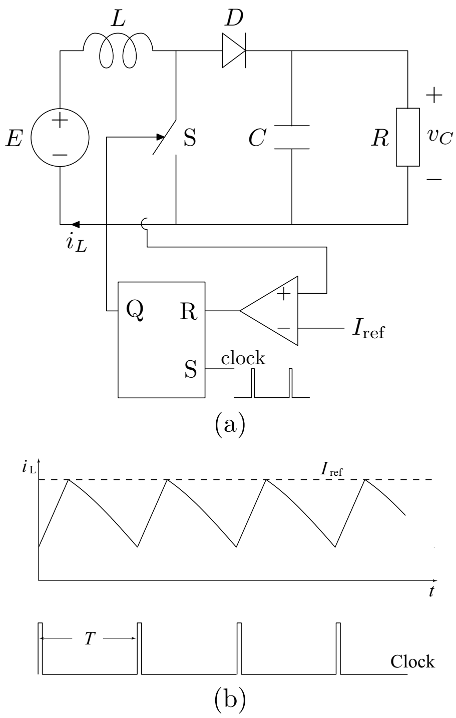

Figure 2: (a) A current-mode controlled boost converter. (b) Inductor current waveform.

ered. By now, most studies are formulated in terms of *piecewise linear* ordinary differential equations, whose solutions are usually written as exponential functions. However, we describe the solution of a switched systems in term of the flow in each space and propose a computation method, independent of the form of solutions. Our method is even capable of *piecewise nonlinear* systems.

2. Method

2.1. Model for switched dynamical systems

The following general considerations can overcome the shortcomings of the previously used models, and hence are applicable to most switched systems.

1. Each subsystem moves in its own space. Spaces are isolated (not divided).
2. There is at least one border in each space. Solutions can only move in one side of the border(s). When it hits the border in its space, switching occurs.
3. Switching action is not border crossing, instead it is "jumping between spaces".

Let us consider a switched dynamical system consisting of m subsystems: $S_1, S_2, \cdots, S_m$.

$$\begin{aligned} S_1 \ &: \quad \dot{x} = f_1(x, \lambda_1), \quad x \in R^n, \\ &\ \ \vdots \\ S_m \ &: \quad \dot{x} = f_m(x, \lambda_m), \quad x \in R^n, \end{aligned} \tag{1}$$

where $\lambda_{1,2,\cdots,m}$ are system parameters. There are m spaces, namely, $M_1, M_2, \cdots, M_m$, corresponding to the m subsystems. The borders in each space are

$$\begin{aligned} B_1 &= \{x \in R^n, \ t \in R \ : \beta_1(.) = 0\} \\ &\ \ \vdots \\ B_m &= \{x \in R^n, \ t \in R \ : \beta_m(.) = 0\} \end{aligned} \tag{2}$$

where $\beta_{1,2,\cdots,m}$ are the switching conditions. Note that β may not be a single equation, since multi-borders are possible for each space and solution jumps to a different space at a border. Thus, $M_k (k = 1, \cdots, m)$ is the portion obtained by removing the part in one side of the "border" $B_k (k = 1, \cdots, m)$,

$$\begin{aligned} M_1 &= \{(x,t) \in R^n \times R : \ \beta_1(.) \geq (\text{or} \leq) 0\} \\ &\ \ \vdots \\ M_m &= \{(x,t) \in R^n \times R : \ \beta_m(.) \geq (\text{or} \leq) 0\} \end{aligned} \tag{3}$$

The solution of the system in M_k is governed by the state equations corresponding to S_k, as given in (1). Suppose the solution in $M_k (k = 1, \cdots, m)$ exists and is given by

$$\begin{aligned} x(t) &= \varphi_1(t, x_0), \quad x(0) = x_0, \quad (x,t) \in M_1 \\ &\ \ \vdots \\ x(t) &= \varphi_m(t, x_0), \quad x(0) = x_0, \quad (x,t) \in M_m \end{aligned} \tag{4}$$

where x_0 is the initial point.

The flow jumps from one space to another when it hits the border. Note that, the jumps between spaces do not lead to jumps of the states; thus, the point at which a flow hits the border can be thought of as the initial point of the successive flow in the next space. This formulation is illustrated in Fig. 3.

If $\beta_k(.) = \beta_{k+1}(.)$, borders B_k and B_{k+1} are identical. In that case, M_k and M_{k+1} are two sides of a common border. This condition can often be observed in systems whose switching is simply controlled by a comparator (e.g., in the voltage-mode controlled buck converter [2]). This is a special case covered by our method.

2.2. Periodic solution and first return map

We now consider a general fundamental solution. Starting at x_0, a solution flow moves in M_1 at first, and then touches the border B_1 at x_1 after time τ_1. Hitting the border causes a jump to M_2. Similarly, hitting border $B_2 \in M_2$ will lead to a jump to M_3.

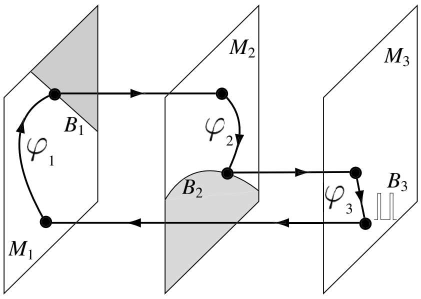

Figure 3: An example illustration of our modelling method consisting three subsystems.

In this way, the solution flow moving in M_k will hit specific border B_k and switch to M_{k+1}. Finally, it will get to M_m, the last space, and then return to M_1 at x_m at time τ_m. Thus, we can define a map as

$$F_m : R^n \mapsto R^n;\ x_0 \mapsto x_m \tag{5}$$

which is similar to the definition of Poincaré map for autonomous systems. We call it First Return Map.

The flow of the fundamental solution described above can be constructed by following equations:

$$\begin{aligned} x_1 &= \varphi_1(\tau_1, x_0) \qquad\qquad (x \in M_1) \\ &\vdots \\ x_m &= \varphi_m(\tau_m - \tau_{m-1}, x_{m-1}) \quad (x \in M_m) \end{aligned} \tag{6}$$

Moreover, because the point where the flow hits the border satisfies the switching conditions (2), we have

$$\begin{aligned} \beta_1(x_1,\ \tau_1) &= 0 \qquad (\text{border } B_1) \\ &\vdots \\ \beta_m(x_m,\ \tau_m) &= 0 \quad (\text{border } B_m) \end{aligned} \tag{7}$$

Because the flow is a fundamental periodic solution only if $x_m = x_0$, we can replace x_m with x_0. Thus, from (6) and (7), we have $m \times n + m$ scalar equations. Meanwhile, unknowns of (6) and (7) are $\{x_0, \cdots, x_{m-1}, \tau_1, \cdots, \tau_m\}$, and the total scalar number is $m \times n + m$. Thus, we can solve the periodic solution using an appropriate numerical method.

2.3. Analysis of stability

Stability of a periodic solution can be determined from the Jacobian of the first return map defined in (5), which is described by $\partial x_m/\partial x_0$. Since the flow is piecewise defined, we rewrite the Jacobian as

$$\frac{\partial x_m}{\partial x_0} = \frac{\partial x_m}{\partial x_{m-1}}\frac{\partial x_{m-1}}{\partial x_{m-2}} \cdots \frac{\partial x_2}{\partial x_1}\frac{\partial x_1}{\partial x_0} = \prod_{i=1}^{m} \frac{\partial x_i}{\partial x_{i-1}} \tag{8}$$

From the ith equation of (6), we get

$$\frac{\partial x_i}{\partial x_{i-1}} = \frac{\partial \varphi_i}{\partial t}\frac{\partial \tau_i}{\partial x_{i-1}} - \frac{\partial \varphi_i}{\partial t}\frac{\partial \tau_{i-1}}{\partial x_{i-1}} + \frac{\partial \varphi_i}{\partial x_{i-1}} \tag{9}$$

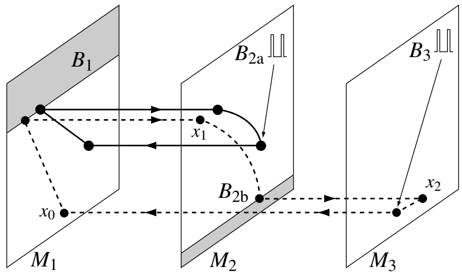

Figure 4: Model of current mode boost converter.

where φ_i and $\partial\varphi_i/\partial x_{i-1}$ denote $\varphi_i(\tau_i - \tau_{i-1}, x_{i-1})$ and $\partial\varphi_i/\partial x|_{x=x_{i-1}}$, respectively. Since x_i and x_{i-1} satisfy the switching conditions: $\beta_i(x_i, \tau_i) = 0$ and $\beta_{i-1}(x_{i-1}, \tau_{i-1}) = 0$, from which we can write

$$\begin{aligned} \frac{\partial \beta_i}{\partial x_{i-1}} &= \frac{\partial \beta_i}{\partial x}\frac{\partial x_i}{\partial x_{i-1}} + \frac{\partial \beta_i}{\partial t}\frac{\partial \tau_i}{\partial x_{i-1}} = 0 \\ \frac{\partial \beta_{i-1}}{\partial x_{i-1}} &= \frac{\partial \beta_{i-1}}{\partial x} + \frac{\partial \beta_{i-1}}{\partial t}\frac{\partial \tau_{i-1}}{\partial x_{i-1}} = 0 \end{aligned} \tag{10}$$

Solving (10), we get $\partial\tau_i/\partial x_{i-1}$ and $\partial\tau_{i-1}/\partial x_{i-1}$. Putting them in (9) yields

$$\frac{\partial x_i}{\partial x_{i-1}} = \frac{f_i(\varphi_i)\dfrac{\partial \beta_{i-1}}{\partial x} + \dfrac{\partial \varphi_i}{\partial x_{i-1}}\dfrac{\partial \beta_{i-1}}{\partial t}}{\dfrac{\partial \beta_i}{\partial t} + f_i(\varphi_i)\dfrac{\partial \beta_i}{\partial x}} \cdot \frac{\dfrac{\partial \beta_i}{\partial t}}{\dfrac{\partial \beta_{i-1}}{\partial t}} \tag{11}$$

where $f_i(\varphi_i)$ represents $\partial\varphi_i/\partial t$. Note that $\partial\varphi_i/\partial x_{i-1}$ can be obtained by solving

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \varphi_i}{\partial x_{i-1}}\right) = \frac{\partial f_i}{\partial x}\left(\frac{\partial \varphi_i}{\partial x_{i-1}}\right), \qquad \left.\frac{\partial \varphi_i}{\partial x_{i-1}}\right|_{t=0} = I_n \tag{12}$$

Thus, substituting (11) into (8) and using an appropriate numerical method, we can calculate the Jacobian of the first return map of a specific periodic solution. Finally, by finding the roots of the characteristic equation, we can determine the stability of the periodic solution.

3. Current-Mode Controlled Boost Converter

Switched dynamical systems can be found in many practical applications. In this section, we will investigate the current-mode boost converter, which has been widely studied by other authors, and we reexamine it here with our new model and method.

3.1. Model description

For the boost converter shown in Fig. 2, which may operate in continuous conduction mode (CCM) and discontinuous conduction mode (DCM) [4]. The model can be represented schematically as in Fig. 4.

We first consider the space M_1. The system with state $x = [v_C, i_L]^T$ is given by

$$S_1:\ \dot{x} = \begin{bmatrix} -1/RC & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} E \qquad (13)$$

In M_1, there is a border B_1 defined by

$$B_1 = \{(x,t) \in R^n \times R : \beta_1 = x(2) - I_{\text{ref}} = 0\}. \quad (14)$$

When x hits B_1, switching occurs and x jumps to M_2, where the system becomes

$$S_2:\ \dot{x} = \begin{bmatrix} -1/RC & 1/C \\ -1/L & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} E. \quad (15)$$

Two borders are available in M_2. One corresponds to the clock signal that resets the switch, the other corresponds to the case when the inductor current drops to zero. We can write the border functions as

$$\begin{aligned} B_{2a} &= \{(x,t) \in R^n \times R : \beta_{2a} = t - kT = 0\} \\ B_{2b} &= \{(x,t) \in R^n \times R : \beta_{2b} = x(2) = 0\} \end{aligned} \qquad (16)$$

Once x touches B_{2a}, it will return to M_1 (as the solid curve indicate). Moreover, if it reaches B_{2b} ahead of B_{2a}, it jumps to M_3. The dynamics in M_3 is described simply by

$$S_3:\ \dot{x} = \begin{bmatrix} -1/RC & 0 \\ 0 & 0 \end{bmatrix} x \qquad (17)$$

When the clock signal arrives, i.e., when x hits B_3,

$$B_3 = \{(x,t) \in R^n \times R : \beta_3 = t - kT = 0\}, \qquad (18)$$

the state will return to M_1 (as shown by the dashed curve). Thus any solution of the current-mode controlled boost converter can be solved by the method introduced in Sec. 2.

3.2. Periodic solutions and bifurcation

We consider the simplest case of the period-1 solution, which is the preferred operation in practice. Basically, the solution under CCM can be constructed as

$$\begin{aligned} &x_1 = \varphi_1(\tau_1, x_0) \\ &x_0 = \varphi_2(\tau_2 - \tau_1, x_1) \\ &x_1(2) - I_{\text{ref}} = 0 \\ &\tau_2 - T = 0 \end{aligned} \qquad (19)$$

Thus, five scalar unknowns x_1, x_2, τ_1 can be obtained. Moreover, by fixing some parameters in Fig. 2 as

$$L = 1.5\text{mH},\ T = 100\mu\text{s},\ I_{\text{ref}} = 0.8\text{A}$$

we get the bifurcation diagram shown in Fig. 5. Three bifurcation curves are given corresponding to different capacitor values. Stable period-1 solutions can be observed on the left-hand side of these curves. As we move across the bifurcation curves from left to right, the solution becomes period-2 and exhibit more complicated behavior, such as border collision and chaos.

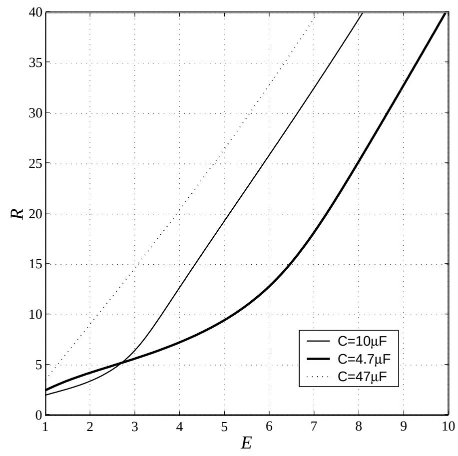

Figure 5: Bifurcation diagram of boost converter in (E, R) plan.

4. Conclusion

In this paper, we propose a general framework for modelling and analyzing switched dynamical systems. The method is capable of modelling most switched systems, and the main features are: (i) Subsystems and their borders are defined in separate spaces. This allows to model the systems without a common border, thus covering a much wider class of switching scenarios. (ii) The method is based on tracking the solution flow, instead of finding the solution. Piecewise linear/nonlinear models can be analyzed by this general method. To demonstrate the modelling approach, we consider the current-mode controlled dc/dc converter and briefly illustrate the application procedure.

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