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Bifurcation and transitional dynamics in three-coupled oscillators with hard type nonlinearity

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Abstract — In this paper, we investigate various bifurcation and related dynamics of certain periodic attractors in a three-coupled oscillator system with hard type nonlinearity. The periodic attractors exist for comparatively large ε (=parameter showing the degree of nonlinearity), and they disappear via saddle-node (S-N) bifurcation when ε becomes small. There exist a heteroclinic and a homoclinic cycle near the bifurcation parameter value for some cases. In such cases, a quasi-periodic attractor appears generally after the S-N bifurcation. In particular, it presents intermittent phenomenon just after the S-N bifurcation. We clarify the existence of the heteroclinic and homoclinic cycles by drawing unstable manifold of saddles on Poincare section, and demonstrate the intermittent phenomenon by simulation.

1. Introduction

In this paper, we investigate various dynamics related to global bifurcation of three coupled oscillators with hard type nonlinearity. At first, we investigate the symmetric case where each oscillator is the same, and next the asymmetric case where it is not the same, and compare the difference between them. For the symmetric case, two symmetric periodic solutions exist, which present the S-N bifurcation at the same value of ε. The saddle and node pairs associated with these two symmetric solutions are connected by unstable manifolds (UM's) of saddles for ε near the bifurcation value to form a heteroclinic cycle at the bifurcation value. Therefore, a switching phenomenon between two periodic solutions can be observed. This is the ultimate form of a quasi-periodic oscillation. In addition to this, there is another periodic solution which also presents the S-N bifurcation. One branch of UM's of this saddle-node pair leads to zero. Therefore, this solution converges to zero after S-N bifurcation.

Next, we investigate the asymmetric case by changing third oscillator's intrinsic frequency. There are no symmetric solutions, and the bifurcation values of three periodic solutions are not the same, and no heteroclinic cycle exists (if the asymmetry exceeds some critical value). In such a case the third solution which bifurcates to zero for the symmetric case forms a cycle connecting a single S-N pair. This becomes a torus after S-N bifurcation. From these facts, we notice that symmetric nature is kept for slightly asymmetric system; namely, the heteroclinic cycle and the resulting switching solution can be observed, but that the heteroclinic cycle breaks when the asymmetry increases, and the resulting dynamics is a torus based on one single S-N pair. We elucidate this fact by drawing unstable manifolds in five dimensional Poincare section.

2. Fundamental equation and the switching phenomenon

The three inductively-coupled oscillators with hard type nonlinearity can be written by the following 6th-order autonomous system:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\varepsilon(1-\beta x_1^2 + x_1^4)x_2 - x_1 + \alpha x_3 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -\varepsilon k_2^2(1-\beta x_3^2 + x_3^4)x_4 - k_2^2(1+\alpha)x_3 + k_2^2\alpha x_1 + k_2^2\alpha x_5 \\
\dot{x}_5 &= x_6 \\
\dot{x}_6 &= -\varepsilon k_3^2(1-\beta x_5^2 + x_5^4)x_6 - k_3^2 x_5 + k_3^2\alpha x_3
\end{aligned} \quad (1)$$

where x_1, x_3 and x_5 denotes the normalized output voltage of the first, second, and third oscillators , x_2, x_4 and x_6 are their derivatives, respectively. The parameter ε>0 shows the degree of nonlinearity. The parameter 0<α<1 is a coupling factor; namely, α=1 means maximum coupling, and α=0 means no coupling.The parameter β controls amplitude of oscillation. The parameter k_2^2 presents the frequency deviation of the first and second oscillators , and k_3^2 presents that of the first and thirdoscillators. Namely, $k_2^2(k_3^2)=1$ means that the first and the second (third) oscillators has an equal intrinsic frequency, and $k_2^2(k_3^2)\neq 1$ means that they have some frequency deviation.

3. Formation and destruction of the heteroclinic cycle

Taking a Poincare section at $x_2 = 0$ in (1), flows in six dimensional phase space become discrete maps in five dimensional phase space $(x_1, x_3, x_4, x_5, x_6)$. Equation (1) is invariant by replacing x_1 by x_5 and x_2 by x_6 for k_2^2 and $k_3^2 = 1$, therefore if there exists a periodic solution P : $(x_1(t), x_2(t), x_3(t), x_4(t), x_5(t), x_6(t))$,
then there exists a symmetric periodic solution P' : $(x_5(t), x_6(t), x_3(t), x_4(t), x_1(t), x_2(t))$ by the above replacement. Fixing α=0.11 and β =3.1, we will present the result of $k_2^2 = k_3^2 = 1$, namely the symmetric system. Fig.1 presents a bifurcation diagram of three periodic solutions for the symmetric case. The upper, middle, and bottom S-N bifurcation curves correspond to the periodic solutions in Figs.2(a),(b) and (c), respectively. Periodic solutions in Figs.2(a) and (c) are symmetric and they form a heteroclinic cycle which bifurcates to a quasi-periodic solution. Right after the S-N bifurcation, the quasi-periodic oscillation takes the form of intermittent or switching oscillation. The periodic solution in Fig.2(b) ceases to zero after S-N bifurcation. The upper and lower S-N bifurcations in Fig.1 occur at the same value of εc =0.374, but the middle one occurs at εc=0.337. Other than these attractors, there exist the same-phase and the reverse phase periodic solutions.

Figure 3 presents the location of nodes and saddles with their UM's obtained by computer simulation for three typical values of ε for the symmetric case. Figure 4 presents a schematic diagram of Fig.3. For ε >0.374 there exist stable nodes N1,N2,N3 and zero. Further, there is a heteroclinic cycle connecting N1 and N3 which bifurcates to be a torus for ε <0.374. The flow stays around the locus of N1 and N3 for a long time and quickly move along the locus unstable manifolds right after the bifurcation, which looks like switching between two attractors. Therefore, we call such a solution the switching solution. For 0.337< ε <0.374 there exist stable nodes N2 and zero and a torus, and for ε <0.337 there exists a stable node zero and a torus. Figure 5 demonstrates a computer simulation of the switching solution. Note that mapped points are distributed on the UM's. In particular they are distributed densely around the locus of N1 and N3. The middle solution associated with N2 goes to zero after the S-N bifurcation, because one of the unstable manifolds of S2 is connected to zero. The structure of nodes and saddles with their UM's shown in Fig.4 is maintained at least for $k_2^2 = 1$ and $1 < k_3^2 < 1.029$.

Next, we will explain the bifurcation diagram for the asymmetric case for $k_2^2 = 1$ and $k_2^2 = 1.03$. Figure 6 presents a bifurcation diagram of four periodic solutions.

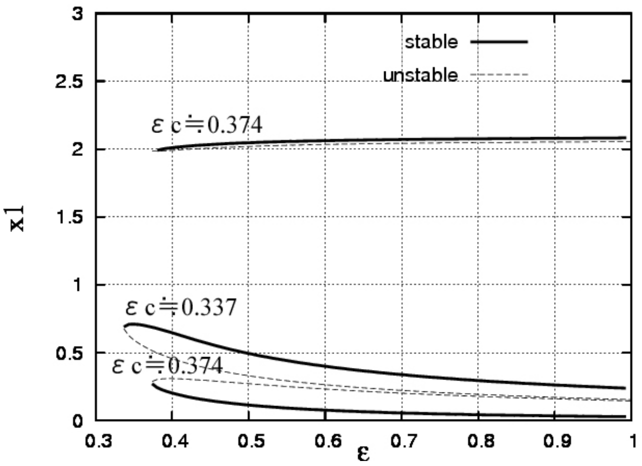


Fig.1: Bifurcation diagram of three periodic solutions for $k_2^2 = k_3^2 = 1$ $\alpha = 0.11$ and $\beta = 3.1$

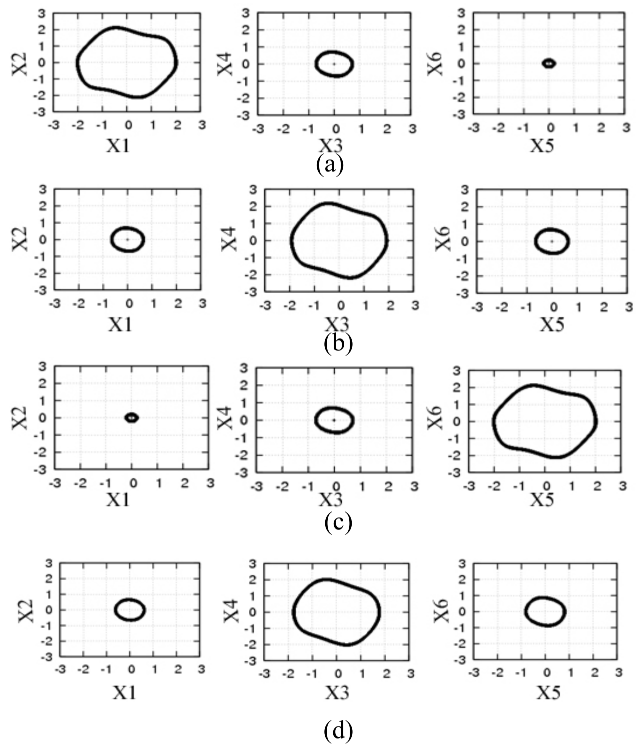
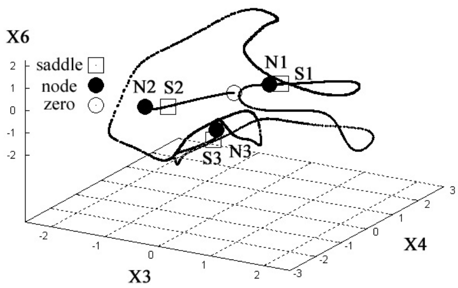
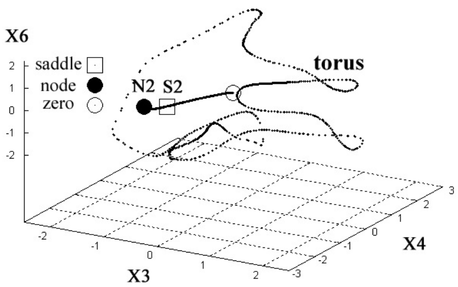


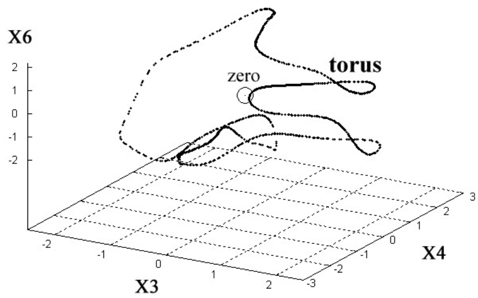
Fig.2 :(a),(b),(c) Three periodic attractors existing for large ε for the symmetric system. Parameters are as follows: $k_2^2 = k_3^2 = 1$ $\alpha = 0.11, \beta = 3.1$ and ε =0.40
(d):A periodic attractor existing for the asymmetric system. Parameters are as follows :$k_2^2 = 1, k_3^2 = 1.03, \alpha = 0.11, \beta = 3.1$ and ε =0.365. This periodic attractor exists for $0.360 < \varepsilon < 0.367$



(a) ε =0.385

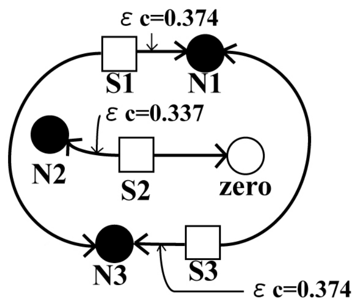


(b) ε =0.37

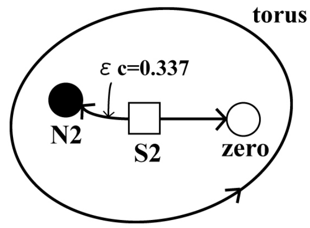


(c) ε =0.33

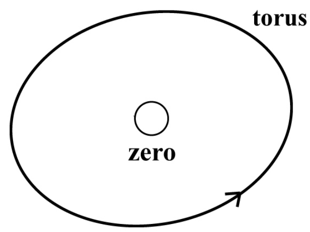
Fig.3: Computer calculation of nodes, saddles and UM's (or torus) for the symmetric case: $k_2^2 = k_3^2 = 1$ $\alpha = 0.11$, $\beta = 3.1$ Projection onto the (x_3, x_4, x_6) - space. □ : saddle ● : node ○ : zero



(a) ε >0.374



(b) 0.337< ε <0.374



(c) ε <0.337

Fig.4: The schematic diagram of nodes, saddles and UM's for the symmetric system

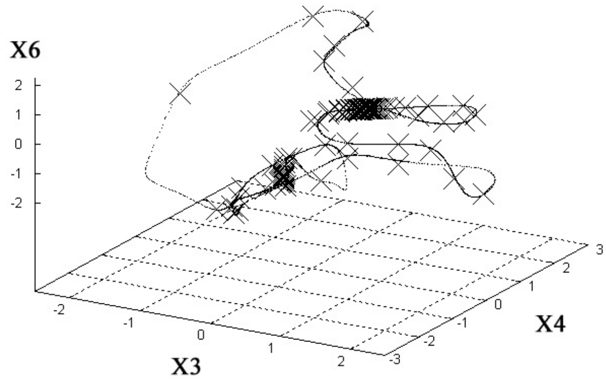


Fig.5: Computer simulation of the switching solution. Parameters are as follows: $k_2^2 = k_3^2 = 1$, $\alpha = 0.11$, $\beta = 3.1$ The cross marks (×) present the Poincare mapped points right after the S-N bifurcation for the switching attractor at ε =0.370. The UM's shows the heteroclinic cycle connecting N1 and N3 at ε = 0.385.

The upper bifurcation curve is the N1-S1 pair corresponding to the periodic solution in Fig.2(a). The second one is the N2-S2 pair corresponding to the periodic solution in Fig.2(b). The bottom one is the N3-S3 pair corresponding to the periodic solution in Fig.2(c). The third one is the N4-S4 pair corresponding to the periodic solution in Fig.2(d), and presents the S-N bifurcation at εc1=0.360. This periodic solution ceases at εc2=0.367, but type of bifurcation is not clear so far. After all, bifurcation points of each solution in the asymmetric case are inconsistent in contrast to the symmetric case.

Figures 7(a)-(f) show schematic diagrams showing the relation of nodes and saddles with their unstable manifolds for the asymmetric case for various values of ε.

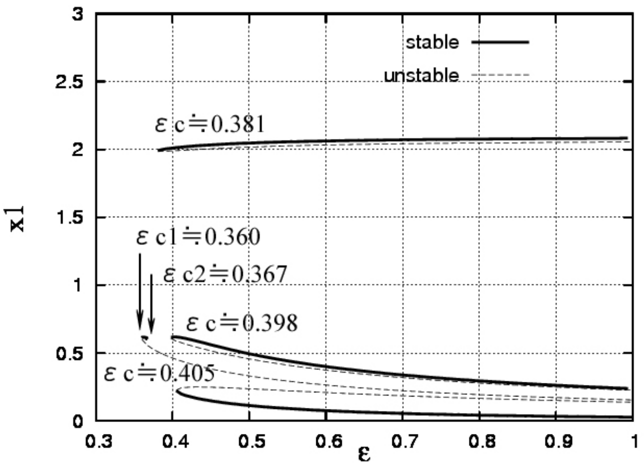
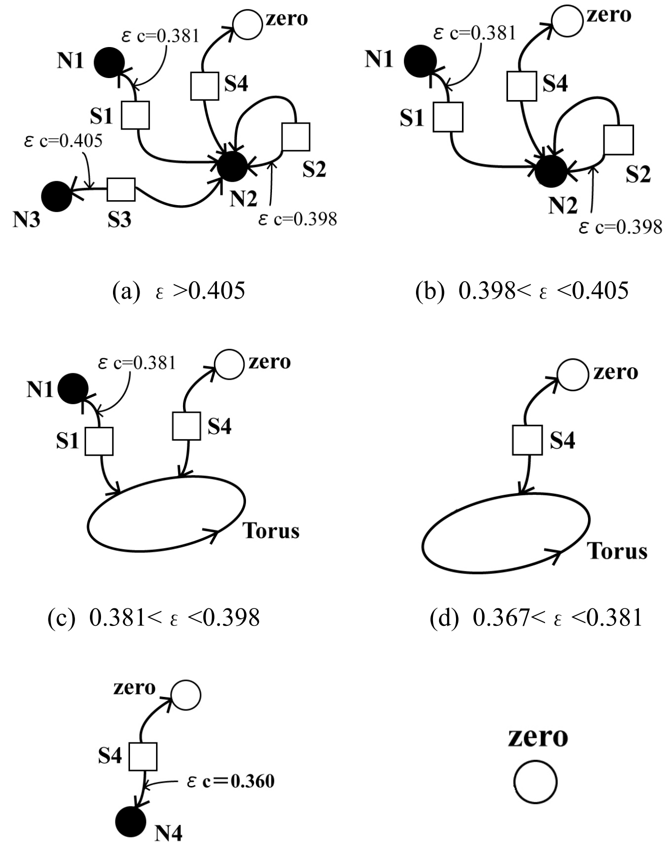


Fig.6: Bifurcation diagram of four periodic solutions for $k_2^2 = 1, k_3^2 = 1.03$ $\alpha = 0.11$ and $\beta = 3.1$ for the asymmetric system

The reason why we do not show a real flow diagram is because it is too complicated to understand. Nodes N1,N2,N3 and N4 corresponds to the periodic oscillations in Figs.2(a),(b),(c) and (d) respectively. For ε>0.405 there exist stable nodes N1,N2,N3 and zero as shown in Fig.7(a). For 0.398<ε<0.405 there exist stable nodes N1,N2 and zero (N3 and S3 disappears via SN-bifurcation) as shown in Fig.7(b). For 0.381<ε<0.398 there exist zero, N1 and a torus (N2 and S2 disappears) as shown in Fig.7(c). For 0.367<ε<0.381 there exist zero and a torus (N1 and S1 disappears) as shown in Fig.7(d). For 0.360<ε<0.367 there exist zero and N4 (torus disappears) as shown in Fig.7(e). For ε<0.36 there exist only zero as shown in Fig.7(f). After all, by changing k_3^2 slightly, the relationship between nodes and saddles with their UM's changes greatly. This is a kind of global bifurcation.

4. Conclusions

In this paper we investigate the global bifurcation and the change of dynamics in terms of ε of three coupled oscillators with hard type nonlinearity. In particular, we compare the results of symmetric and asymmetric systems. Namely, for the symmetric system there is a heteroclinic cycle which bifurcates to be a torus. This torus exists for small ε . In addition, switching dynamics of two periodic solutions can be observed right after the S-N bifurcation. For the asymmetric system there is a homoclinic cycle which bifurcates to be a torus. However, the torus disappears for small ε . The dynamics is very sensitive to the changes of ε as shown in the schematic diagram of Figs.4 and 7. In the future, we will investigate for many combinations of k_2^2 and k_3^2



(a) ε >0.405 (b) 0.398< ε <0.405

(c) 0.381< ε <0.398 (d) 0.367< ε <0.381

(e) 0.360< ε <0.367 (f) ε <0.360

Fig.7 : The schematic diagram of nodes, saddles and UM's for $k_2^2 = 1$, $k_3^2 = 1.03$ for the asymmetric system

References

[1] Y. Aruga and T. Endo, "Transient dynamics observed in strongly nonlinear mutually-coupled oscillators", NOLTA 2002 , pp.135-138 (2002).

[2] Y. Aruga and T. Endo, "Transitional dynamics and chaos in coupled oscillator systems", Trans. IEICE(A), vol. J86-A,No.5, pp.559-568 (2003)