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Bifurcation and transitional dynamics in asymmetrical two-coupled oscillators with hard type nonlinearity

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Abstract- In this paper we deal with various global bifurcations and related change of dynamics including the transitional phenomenon in asymmetrical two-coupled oscillators with hard type nonlinearity. There exist periodic attractors for comparatively large ε (= a parameter showing the degree of nonlinearity), but for small ε, they disappear because of saddle-node bifurcation. We draw nodes and saddles with their unstable manifold near to the bifurcation point on Poincare section for various values of k^2 (= a parameter showing the deviation between two oscillators' intrinsic frequencies). We found that the unstable manifold changes in various ways with variation of parameters around the bifurcation point, and therefore the associated dynamics changes drastically.

1. Introduction

In [1][2], we focused our attention on the transitional phenomenon of a switching attractor in identical (hence symmetric) two-coupled oscillator systems with hard type nonlinearity. We clarified that the key mechanism existed in a heteroclinic cycle connecting two degenerate saddles and nodes which was formed at the bifurcation point. In this paper, we deal with various dynamics including transitional phenomenon in asymmetrical two-coupled oscillators with hard type nonlinearity. In particular, we investigate how the heteroclinic cycle in symmetrical system disappears and new connections of nodes and saddles appear when the system becomes asymmetric. Accordingly, we draw various diagrams of nodes, saddles and their unstable manifolds (UM's) in terms of frequency deviation k^2 in order to observe the change of behavior of UM's with the value of k^2. It is very natural and realistic to assume that each oscillator has a slightly different intrinsic oscillation frequency, because inevitable errors between two oscillators are left in practice. We focus our notice on the connection between nodes and saddles by UM's to elucidate the whole dynamics for such an asymmetric system. As a result, it becomes clear that the system presents various connections associated with the degree of frequency deviation.

2. Fundamental equation and the switching phenomenon

The asymmetric two inductively-coupled oscillators with hard type nonlinearity can be written by the following 4[th]-order autonomous system:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\varepsilon(1-\beta x_1^2 + x_1^4)x_2 - x_1 + \alpha x_3 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = -\varepsilon k^2(1-\beta x_3^2 + x_3^4)x_4 - k^2 x_3 + \alpha k^2 x_1 \end{cases} \tag{1}$$

where x_1 denotes the normalized output voltage of one oscillator, x_2 is its derivative, and where x_3 denotes the normalized output voltage of the other oscillator, x_4 is its derivative. The parameter $\varepsilon > 0$ shows the degree of nonlinearity. The parameter $0 < \alpha < 1$ is a coupling factor; namely, $\alpha = 1$ means maximum coupling, and $\alpha = 0$ means no coupling. The parameter β controls amplitude of oscillation. The parameter k^2 presents the frequency deviation of two oscillators; namely, $k^2 = 1$ means that two oscillators has an equal intrinsic frequency, and $k^2 \neq 1$ means that they have some frequency deviation. This system has three attractors in general for small ε; namely, the same-phase (periodic) attractor, the reverse-phase (periodic) attractor, and the double-mode (quasi-periodic) attractor. When ε becomes large, the double-mode attractor becomes two periodic attractors as shown later. In particular, for ε smaller than, but close to the bifurcation point, one can observe the switching phenomenon of these attractors[1] which is the ultimate form of the double-mode attractor. The reason for this switching solution is a formation of a heteroclinic cycle associated with the degenerate saddles. In this paper, we investigate global bifurcation of this heteroclinic cycle for $k^2 \neq 1$.

3. Global bifurcation of the heteroclinic cycle

Taking a Poincare section at $x_2 = 0$ in (1), flows in four dimensional phase space become discrete maps in three dimensional phase space (x_1, x_3, x_4). First of all, we will present the result of $k^2 = 1$ for review. Figure 1 presents a bifurcation diagram for $k^2 = 1.0$, $\alpha = 0.1$ and $\beta = 3.1$ of two periodic solutions which bifurcate to be a quasi-periodic oscillation for small ε. Equation (1) is invariant by replacing x_1 by x_3 and x_2 by x_4 for $k^2 = 1$, therefore, if there exists a periodic solution P: ($x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$), then there exists another periodic solution P': $(x_3(t), x_4(t), x_1(t), x_2(t))$ by the above replacement. This means that two saddle-node bifurcations occur at the same value of $\varepsilon = 0.449$ in Fig.1[1]. In this case, the same-phase and the reverse-phase periodic solutions are stable, and no bifurcation occurs for $0 < \varepsilon < 0.6$, therefore they are omitted. Figure 2 shows nodes (corresponding to the stable periodic attractors) and the associated saddles, of which UM's connect two nodes and saddles for $\varepsilon = 0.449$. Note a cycle connecting two (almost degenerate) saddles, which is approximately a heteroclinic cycle. When ε becomes a bit smaller, the nodes and the saddles disappear but their "locus" exist. The same is said for the UM's. Therefore, the flow for example, for $\varepsilon = 0.447$ behaves as follows. 1) Flow stays around the locus of one node for a long time, and 2) quickly moves along the locus of UM, and 3) it stays again around the locus of the other node for a long time, and 4) moves quickly along the locus of UM, vice versa. We call this the switching phenomenon. This flow behavior is verified by the Poincare mapped points of the switching attractor in Fig.2. The triangular mark and the square mark show the reverse and the same phase periodic attractors, respectively. In this case, they have no relation to the switching solution.

Figure 3 shows the bifurcation diagram of the periodic solution for the asymmetric case for $k^2 = 1.033, \alpha = 0.1$ and $\beta = 3.1$. The S-N bifurcation point for the upper periodic solution is $\varepsilon = 0.445$, and that for the lower one is $\varepsilon = 0.438$. Figure 4 shows two nodes, two saddles and so on. Notice that there still exists an approximate heteroclinic cycle in spite of asymmetry, and therefore, one can observe the switching phenomenon for smaller ε[2]. The switching phenomenon is verified by Poincare mapped points in Fig.4 for $\varepsilon = 0.437$. The reason why there are more mapped points on the lower UM than those on the upper UM, is that the deviation from the bifurcation point is smaller for the lower UM ($\Delta\varepsilon = 0.001$) than for the upper UM ($\Delta\varepsilon = 0.008$). When $0.438 < \varepsilon < 0.445$, the saddle-node pair denoted by A is lost, but that denoted by B still exists. Therefore, if one gives an initial condition around A, the flow follows the locus of the upper UM and converges to the node in B. For $\varepsilon < 0.438$, the upper and lower unstable manifolds both disappear, therefore the flow switches between A and B.

Figure 5 shows the bifurcation diagram of the periodic solution for the asymmetric case for $k^2 = 1.035$, $\alpha = 0.1$ and $\beta = 3.1$. The S-N bifurcation point for the upper solution is $\varepsilon = 0.444$, and that for the lower one is $\varepsilon = 0.437$. Figure 6 shows the relationship between nodes and saddles and so on. Note that there is no heteroclinic cycle; namely, the upper UM connects two nodes in A and B, while the lower UM connects the node in B and R (=stable reverse-phase solution). Therefore, starting around the initial condition in A, the flow reaches the node in B for $0.437 < \varepsilon < 0.444$ and stays there forever. Further, for $\varepsilon < 0.437$ the flow stays in B for some time but eventually moves toward R. In the same way, the resulting transitional dynamics change with the value of k^2 due to

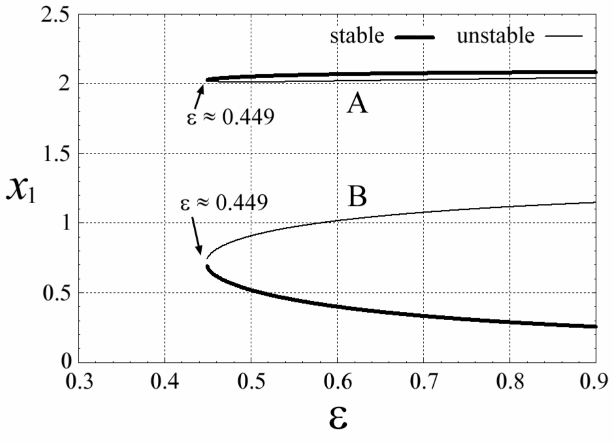

Figure 1: Bifurcation diagram of two symmetric periodic solutions for the symmetric system : $k^2 = 1.0$, $\alpha = 0.1$ and $\beta = 3.1$. The upper trace corresponds to the periodic attractor associated with the initial condition ($x_1(0)$, $x_2(0)$, $x_3(0)$, $x_4(0)$)=(2,0,0,0). The lower trace corresponds to the periodic attractor associated with (0,0,2,0).

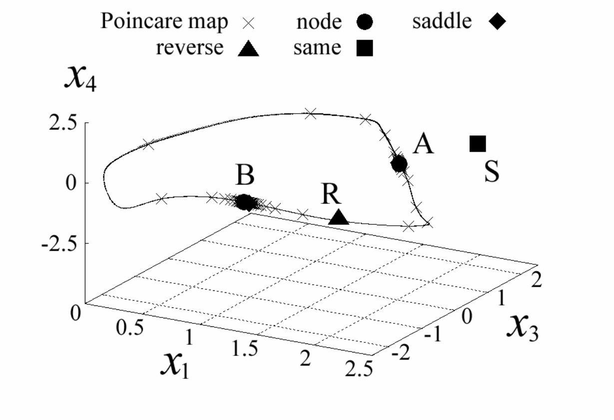

Figure 2: Nodes and saddles with their UM's for $k^2 = 1.0$, $\alpha = 0.1$ and $\beta = 3.1$. The UM is drawn for $\varepsilon = 0.449$. The cross marks ($\times$) present the Poincare mapped points of the switching attractor at $\varepsilon = 0.447$.

[1] The precise value of S-N bifurcation is a bit smaller than this value. Same is said for Fig.3 and Fig.5 and so on.

[2] In fact, this is not a heteroclinic cycle but a homoclinic cycle around $\varepsilon = 0.438$. One of the UM of the homoclinic cycle passes through near the locus of a degenerate saddle in A. Therefore it looks like a quasi-heteroclinic cycle.

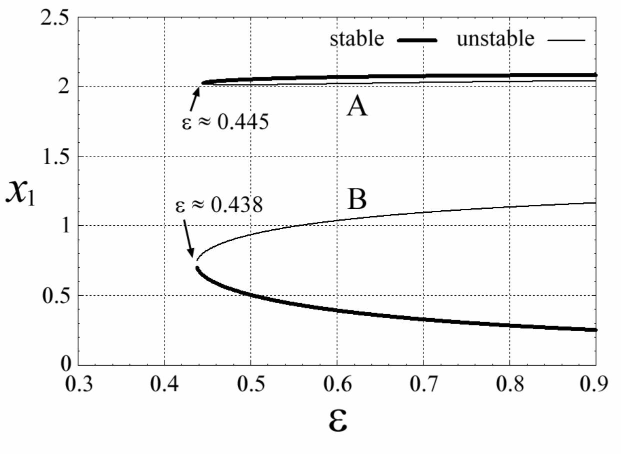

Figure 3: Bifurcation diagram of two periodic solutions for the asymmetric system : $k^2 = 1.033$, $\alpha = 0.1$ and $\beta = 3.1$. The periodic solutions corresponding to the upper and lower traces are explained in Fig.1.

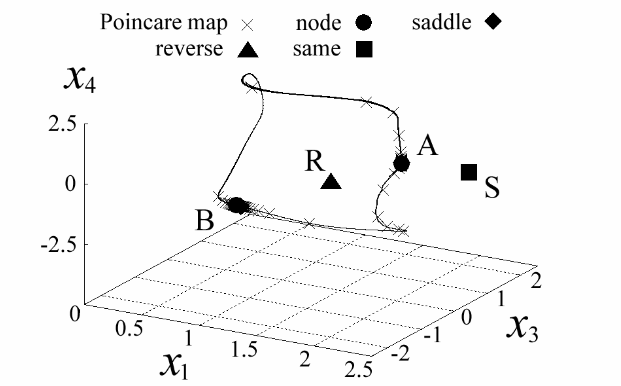

Figure 4: Relationship between nodes and saddles for $k^2 = 1.033$, $\alpha = 0.1$ and $\beta = 3.1$. The upper unstable manifold is drawn for $\varepsilon = 0.445$, and the lower one is drawn for $\varepsilon = 0.438$. The cross marks ($\times$) present the Poincare mapped points for the switching attractor at $\varepsilon = 0.437$.

different behavior of UM's. The behavior of the flow starting around A at $\varepsilon = 0.436$ is shown by cross marks in Fig.6. Observe that the points move along the upper UM quickly ($\Delta\varepsilon = 0.008$) and stay on the locus of node in B for a long time and again move along the lower UM relatively slowly ($\Delta\varepsilon = 0.001$) and converge to R.

Figure 7 shows the relationship between nodes and saddles and so on at $k^2 = 1.048$. For the S_1-N_1 pair A, the S-N bifurcation occurs approximately for $\varepsilon = 0.440$ and for the S_2-N_2 pair B, it occurs approximately for $\varepsilon = 0.431$. For $0.431 < \varepsilon < 0.440$, the flow starting around A follows the locus of UM and converges to N_2 in B. For $\varepsilon < 0.431$, the flow stays around the locus of S_2-N_2 in B and around R (=unstable reverse-phase solution) for a long time and moves quickly along the locus of the UM.

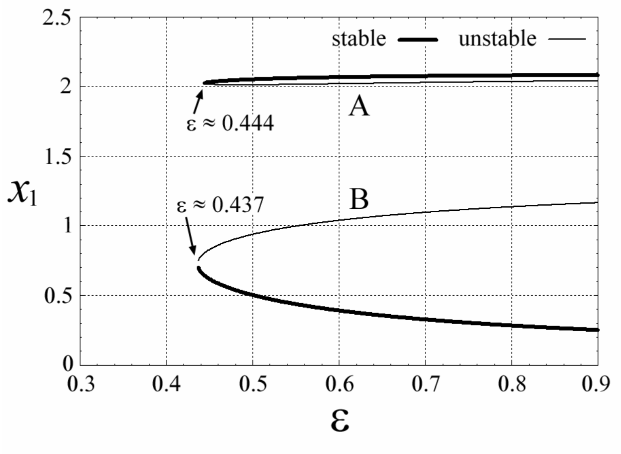

Figure 5: Bifurcation diagram of two periodic solutions for the asymmetric system : $k^2 = 1.035$, $\alpha = 0.1$ and $\beta = 3.1$. The periodic solutions corresponding to upper and lower traces are explained in Fig.1.

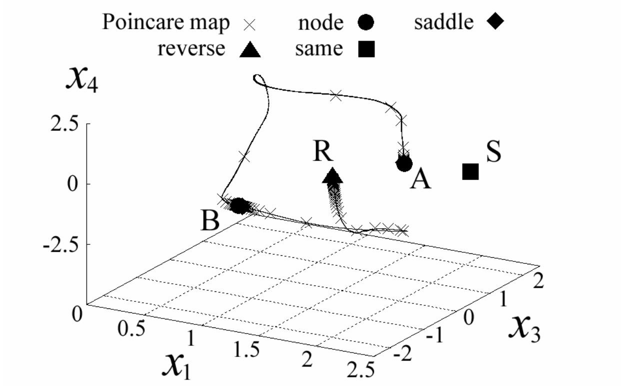

Figure 6: Relationship between nodes and saddles for $k^2 = 1.035$, $\alpha = 0.1$ and $\beta = 3.1$. The upper UM is drawn for $\varepsilon = 0.444$ and the lower UM is drawn for $\varepsilon = 0.437$. The cross marks ($\times$) present the Poincare mapped points for the initial condition around A at $\varepsilon = 0.436$. The S is unstable and the R is stable.

Figure 8 shows the relationship between nodes and saddles and so on at $k^2 = 1.052$. For the S_1-N_1 pair A, the S-N bifurcation occurs approximately for $\varepsilon = 0.439$ and for the S_2-N_2 pair B, it occurs approximately for $\varepsilon = 0.428$. In this case the heteroclinic cycle in Fig.1 is divided in two homoclinic cycles. Therefore, for the initial condition around A at $\varepsilon = 0.438$, the mapped points draw the homoclinic cycle associated with A, namely a quasi-periodic oscillation can be observed. Further, for the initial condition around B at $\varepsilon = 0.427$, the mapped points draw the homoclinic cycle associated with B, namely another quasi-periodic oscillation can be observed.

Figure 9 summarizes the relationship between nodes, saddles and UM's for various value of k^2. In Fig.9 (a) for

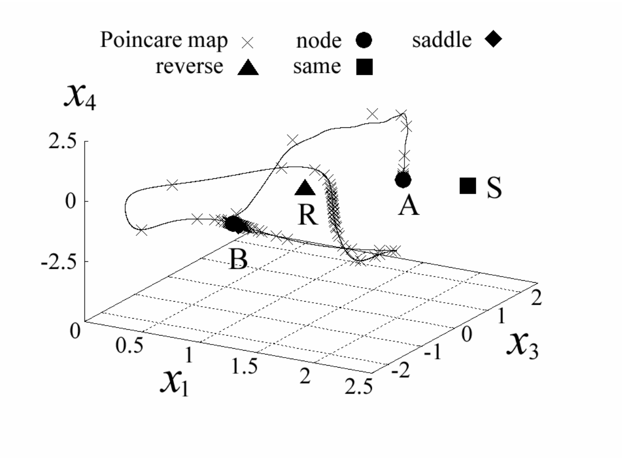

Figure 7: Relationship between nodes and saddles for $k^2 = 1.048$, $\alpha = 0.1$ and $\beta = 3.1$. The upper UM is drawn for $\varepsilon = 0.440$ and the lower UM is drawn for $\varepsilon = 0.431$. The cross marks ($\times$) present the Poincare mapped points for the initial condition around A at $\varepsilon = 0.429$. The S and R are unstable.

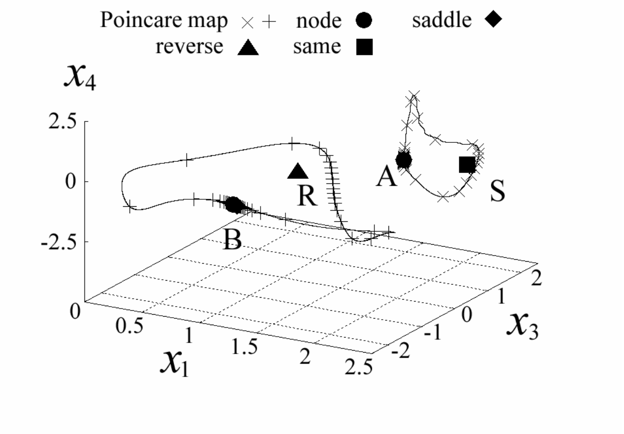

Figure 8: Relationship between nodes and saddles for $k^2 = 1.052$, $\alpha = 0.1$ and $\beta = 3.1$. The right-hand side UM is drawn for $\varepsilon = 0.439$ and the left-hand side UM is drawn for $\varepsilon = 0.428$. The cross marks ($\times$) present the Poincare mapped points for the initial condition around A at $\varepsilon = 0.438$ and the other cross marks ($+$) present them for the initial condition around B at $\varepsilon = 0.427$. The S and R are unstable.

$k^2 = 1$, the S_1-N_1 and S_2-N_2 bifurcation points are the same, therefore, there exists a real heteroclinic cycle. However, in Fig.9 (a') for $1 < k^2 < 1.034$, the UM of S_2 goes to N_2, but on the way it approaches the locus of S_1-N_1 and stays there for a long time. We call such a homoclinic cycle a quasi-heteroclinic cycle. It is interesting that manner of connection of nodes and saddles by UM's changes a lot with variation of k^2, and hence transitional dynamics change with k^2.

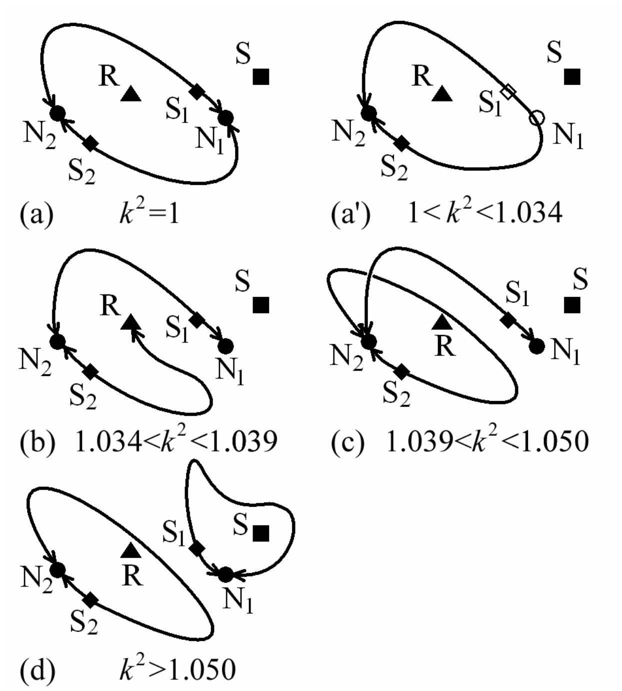

Figure 9: Schematic diagrams for various values of k^2. $N_1(N_2)$, $S_1(S_2)$, R and S denote nodes, saddles, the reverse-phase periodic solution and the same-phase periodic solution, respectively.

4. Conclusions

In this paper, we elucidate the global bifurcation associated with the unstable manifold of saddles for the periodic solution bifurcated from a quasi-periodic attractor in asymmetric two-coupled oscillators with hard type nonlinearity. In the future, we will investigate more thorough bifurcation of this system.

References

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[2] Y. Aruga and T. Endo, "Transitional dynamics and chaos in coupled oscillator systems", Tranc. IEICE (A), vol. J86-A, No.5, pp.559-568, 2003.