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Fractal in Brain Dynamics

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Abstract—In this paper a theory developed by the author is reviewed for brain dynamics of recurrent neural networks. Recently, ”Hybrid Dynamical Systems” have attracted considerable attention in automatic control domain. The hybrid dynamical system is defined by a continuous dynamical system discretely switched by external temporal inputs. The theory suggests that the dynamics of continuous-time recurrent neural networks, which is stochastically excited by external temporal inputs, is generally characterized by a set of continuous trajectories with a fractal-like structure in hyper-cylindrical phase space.

1. Introduction

A *Hybrid Dynamical System* (HDS) [1] is a dynamical system that involves an interaction of discrete and continuous dynamics, e.g., a continuous-time device controlled by a sliding mode controller [2]. Control strategies introduced for HDS have been applied to various design problems for realizing desired behavior in power plants [3], chemical plants [4], etc.

The dynamical evolution of HDS can be described by a differential equation involving a number of vector fields that are switched one after another [5]. It has been shown that HDS displays very complex behaviors such as chaotic behavior [6]. Branicky has presented a numerical experiment for HDS described by a simple linear equation and shown that the state of the system moves around on the Sierpinski gasket, a very well-known fractal set. This result suggests that the fractals may universally appear in some classes of HDS [7]. In this paper a theory for continuous recurrent neural networks with temporal inputs are reviewed from hybrid dynamical systems point of view.

We focus on dissipative, continuous, and non-autonomous recurrent neural networks defined by the following ordinary differential equations:

$$\dot{x} = f(x, t), \qquad (1)$$
$$x \in R^N,$$

where x, t and f are state, time, and vector field, respectively. Equation (1) implies that the vector field depends on time. In general, this suggests that a network is influenced by other systems. To emphasize that the vector fields depend on time throughout the input $I(t)$, we rewrite Eq. (1) as follows:

$$\dot{x} = f(x, I(t)), \qquad (2)$$
$$x, I \in R^N.$$

2. Dynamics with Periodic Inputs

We will begin by considering a dynamics with a periodic input:

$$I(t) = I(t + T),$$

where T is the period of the input. The vector field f is also periodic with the same period T:

$$f(t) = f(t + T).$$

Introducing the angular variable $\theta = \frac{2\pi}{T} t \bmod 2\pi$ and new state variable $y = (x, \theta)$, we can transform the non-autonomous system expressed by Eq. (2) into the following autonomous system:

$$\dot{y} = f_I(y), \qquad (3)$$
$$y \in R^N \times S^1.$$

The vector field f_I is defined on a manifold $\mathcal{M} : R^N \times S^1$ that is a hyper-cylindrical space. In other words, Eq. (3) expresses a continuous dynamical system D_c defined by the manifold $\mathcal{M}$ and the vector field f_I:

$$D_c = (\mathcal{M}, f_I). \qquad (4)$$

In the hyper-cylindrical space $\mathcal{M}$, we can define the Poincaré section:

$$\Sigma = \left\{ (x, \theta) \in R^N \times S^1 | \theta = 2\pi \right\},$$

where a trajectory starting from an initial state at $\theta = 0$ returns at $\theta = 2\pi$. On the section Σ, a mapping can be defined which transforms a state x_τ to another state $x_{\tau+1}$ after interval T:

$$x_{\tau+1} = g_I(x_\tau), \qquad (5)$$
$$x_\tau \in R^N,$$

where g_I is an iterated function. In other words, Eq. (5) expresses a discrete dynamical system D_d defined by the manifold Σ and the iterated function g_I:

$$D_d = (\Sigma, g_I). \qquad (6)$$

We can summarize the dynamics with a periodic input as follows. The periodic input I defines two dynamical systems, a continuous one D_c and a discrete one D_d defined by Eqs. (4) and (6), respectively. In order to emphasize the relation among I, D_c and D_d, we use the following schematic expression:

$$I \to D_c \to D_d. \quad (7)$$

3. Dynamics with Switching Inputs

3.1. A Set of Inputs

In this section, we consider a dynamics in which plural input patterns are stochastically fed into the system one after the other. Let us suppose that *each input is one period of a periodic function*. For example, we can define the periodic function by the following Fourier series:

$$I(t) = \frac{a_0}{2} + \sum_{m=1}^{M} (a_m \cos \frac{2\pi m}{T} t + b_m \sin \frac{2\pi m}{T} t), \quad (8)$$

where a_0, a_m, $b_m \in R^N$ are vectors for Fourier coefficients, and T is the period. The set of these parameters defines the *input space*:

$$\mathcal{I} = \left\{ a_0, \{a_m, b_m\}_{m=1}^{M}, T \right\},$$

$$\mathcal{I} : R^{N+2\times N\times M+1}.$$

Within this space, an arbitrary point represents an external temporal input. We consider the input as a set $\{I_l\}_{l=1}^{L}$ of time functions I_l sampled on the parameterized space $\mathcal{I}$. In the following sections, we abbreviate subscripts and express individual sets as $\{\cdot\}$ for simplicity.

3.2. Two Sets of Dynamical Systems

Much as in the case of periodic input, we can define two sets of dynamical systems corresponding to the set $\{I_l\}$. One is the set of continuous dynamical systems:

$$\{D_{cl}\} = (\mathcal{M}, \{f_l\}), \quad (9)$$

where $\{f_l\}$ is the set of vector fields on the hyper-cylindrical space $\mathcal{M}$. The other is the set of discrete ones:

$$\{D_{dl}\} = (\Sigma, \{g_l\}), \quad (10)$$

where $\{g_l\}$ is the set of iterated functions on the global Poincaré section Σ. We also use the following schematic expression, which is similar to expression (7):

$$\{I_l\} \to \{D_{cl}\} \to \{D_{dl}\}. \quad (11)$$

3.3. Excited Attractor

In this paper, we are considering a continuous dynamical system that is dissipative and has an attractor for a periodic input. When an input pattern is fed into the system repeatedly, i.e, in the case of periodic input, a trajectory converges to an attractor. But how do we describe the dynamics when the inputs are switched stochastically? Even for an input with finite interval, we can assume an attractor corresponding to a periodic input with infinite interval. We call this an *excited attractor* in order to emphasize that the attractor is excited by the external input. Although a trajectory tends to converge to a corresponding excited attractor, the trajectory cannot reach the excited attractor due to the finite time interval. If the next input is the same as the previous one, the trajectory again goes toward the same excited attractor. If the next input is different from the previous one, the trajectory changes its direction and goes toward an excited attractor distinct from the previous one. Continuing this process, the trajectory takes a transient path to the excited attractors. Intuitively, the trajectory will be spread out around excited attractors in the hyper-cylindrical phase space $\mathcal{M}$. How, then, do we characterize the properties of the transient trajectory?

4. Fractal Transition

4.1. Iterated Function System

In the following two sections, we focus on the set $\{g_l\}$ of iterated functions on the global Poincaré section Σ.

4.1.1. Hutchinson's Theory

Hutchinson [8] has proved that a set $\{h_l\}$ of iterated functions, which are not limited on the Poincaé section, defines a unique and invariant set C that satisfies the following equation:

$$C = \bigcup_{l=1}^{L} h_l(C), \quad (12)$$

where

$$\bigcup_{l=1}^{L} h_l(C) = h_1(C) \cup h_2(C) \cup \cdots \cup h_L(C),$$

and

$$h_l(C) = \bigcup_{x\in C} h_l(x).$$

Such an invariant set C is often a fractal or sometimes used as a mathematical definition of various fractals.

A sufficient condition for satisfying Eq. (12) is the *contraction* property of h_l for all $l = 1, 2, ..., L$. The contraction for h_l is definitely defined by the Lipschitz constant $Lip(h_l)$:

$$Lip(h_l) = \sup_{x_i \neq x_j} \frac{d(h_l(x_i), h_l(x_j))}{d(x_i, x_j)}, \quad (13)$$

where d is a distance on a metric space. When

$$Lip(h_l) < 1,$$

the map $h_l : x \to x$ is called the *contraction* or the *contraction map*.

4.1.2. Iterated Function System with Probabilities

Barnsley has named a set $\{h_l\}$ as the IFS (*Iterated Function System*) [9]. He introduced the *IFS with probabilities* as follows:

$$(\{h_l\}, \{p_l\}), \quad (14)$$

where $\{p_l\}$ is a set of probabilities corresponding to $\{h_l\}$.

Based on the IFS with probabilities, he proposed an alternative method for constructing the invariant set C that satisfies Eq. (12). The iterated functions h_l are switched with probabilities p_l for $l = 1, 2, ..., L$ as follows. Choose an initial point and then choose recursively and independently $x_{\tau+1} \in \{h_1(x_\tau), h_2(x_\tau), ..., h_L(x_\tau)\}$ for $\tau = 0, 1, 2, ..., \infty$, where the probability of the event $x_{\tau+1} = h_l(x_\tau)$ is p_l. Thus a sequence constructs a set $\{x_n\}_{n=0}^{\infty}$. Using Hutchinson's theory, Barnsley has shown that the set $\{x_n\}_{n=0}^{\infty}$ constructed by random sequence, and here assumed to have equal probability, "converges to" the set C defined by Eq. (12) when all iterated functions are the contractions. The set $\{x_n\}_{n=0}^{\infty}$ is thus an approximation of C.

4.2. Vector Field System

We are now ready to consider the trajectory of continuous dissipative dynamical systems excited by the temporal inputs. When the inputs I_l are stochastically fed into the system one after another, the vector fields f_l and the iterated functions g_l are also stochastically switched as explained in Sec. 3. To emphasize the relation among the set $\{I_l\}$, $\{f_l\}$ and $\{g_l\}$, we use the following schematic expression instead of expression (11):

$$\{I_l\} \to \{f_l\} \to \{g_l\}. \quad (15)$$

We call the set $\{f_l\}$ the *Vector Field System* (VFS), which is similar to the *Iterated Function System* (IFS) for the set $\{g_l\}$. The discrete dynamics on the Poincaré section Σ correspond to the random iteration algorithm using the IFS with probabilities. That is, when all iterated functions g_l are the contractions, the state x_τ on the Poincaré section approximately changes on the invariant set C defined by Eq. (12) after sufficient random iterations. The property of the set C having the fractal structure affects the trajectory in the hyper-cylindrical phase space $\mathcal{M}$.

The trajectory set $\Gamma(C)$ corresponding to the input set $\{I_l\}$ is obtained by the union of the trajectory set $\gamma_l(C)$ for each input I_l:

$$\begin{aligned} \Gamma(C) &= \bigcup_{l=1}^{L} \gamma_l(C), \\ &= \gamma_1(C) \cup \gamma_2(C) \cup \cdots \cup \gamma_L(C). \quad (16) \end{aligned}$$

We can conclude that the dissipative dynamical systems excited by plural temporal inputs are characterized by the trajectory set $\Gamma(C)$ starting from the initial set C defined by Eq. (12). All of the trajectories are considered to represent the transition between the excited attractors. We call this the *fractal transition between the excited attractors*. At this point, we should emphasize that the contraction property of iterated functions defined on the Poincaré section is a *sufficient*, but not *necessary*, condition for the fractal transition.

5. Discussion

In this paper a theoretical framework is reviewed for a recurrent neural networks stochastically excited by external temporal inputs. In this section we discuss some related works. More general theory has been presented in order to model complex systems that interact strongly with other systems. It has been revealed that these dynamics are generally characterized by fractals when the iterated functions are not the contractions [10]. The hierarchical structure of fractals and the noise effect of inputs have been investigated [11]. The fractals generated by switching vector fields have been observed in different domains such as a forced damped oscillator [12], an electronic circuit [13], artificial neural networks [14], and human behavior [15]. Closure of the fractals in both linear [16] and non-linear systems [17] has been also presented. A set of attractors obtained by periodic inputs can approximate trajectories of fractals [18]. These works show that fractals are indispensable for understanding of dynamics observed in the brain.

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