2004 International Symposium on Nonlinear
Theory and its Applications (NOLTA2004)
Fukuoka, Japan, Nov. 29 - Dec. 3, 2004

Numerical Method of Proving Existence of Solution for Nonlinear ODE using Green's Function Expression

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Abstract—In this paper, a method is proposed to prove the existence of solutions for nonlinear ordinary equations on an essencially bounded functional space. Green's Function Expression is used to prevent from overestimating integral arithmetic for set functions by Affine Arithmetic.

1. Introduction

Many methods have been proposed to prove the existence of solutions for nonlinear ordinary differential equations. Especially, one of the authors, Oishi, proposed a method to prove the existence of solutions for nonlinear ordinary differential equations on a continuous functional space. This method uses Krawczyk's operator on a functional space [1]. Krawczyk-like operator is constructed from Newton operator using Mean Value Theorem. This operates is from an interval on a functional space to an interval on a functional space. In order to calculate the image of Krawczyk-like operator, Interval Arithmetic on the functional space is used. We have used Affine Arithmetic instead of Interval Arithmetic in order to avoid the explosion of interval as the result of calculation [4].

In this paper, we shall revise the form of Krawczyk-like operator for computational accuracy than for computational complexity using Green's function expression.

2. Preliminaries

In this section, we introduce the theorem to prove the existence of the solution for nonlinear ordinary differential equations. we also introduce some arithmetic and inclusion of Heviside's step function.

2.1. Formalization to Operator Equation

In this subsection, we formalize nonlinear ordinary equation to an operator equation on a Banach space.

We consider the following nonlinear boundary value problem of a system of first order real differential equations:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(x), \\ g(x) = \displaystyle\sum_{k=1}^{n} B_k x(t_k) + b = 0 \quad (t \in J = [0,1]), \end{cases} \tag{1}$$

where x is a n-dimensional vector valued function on J, $f(x)$ is a n-dimensional vector valued function, B_ks are n-dimensional matrices, b is a n-dimensional vector and $0 = t_1 < t_2 < \cdots < t_n = 1$.

In the following, we assume that an approximate solution $c(t)$ is given for the problem (1). We also assume that it is a step function. Under these assumptions, we will present a sufficient condition on which the problem has an exact solution in a domain containing an approximate solution $c(t)$. Let X be the Banach space of real n-dimensional vector valued functions $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))$ which is essentially bounded on the interval J with the maximum norm

$$\|x\|_\infty = \max_{1 \leqq i \leqq n} \max_{t \in J} |x_i(t)|. \tag{2}$$

In the following, $\|\cdot\|$ means $\|\cdot\|_\infty$. Let $Y = X \times R^n$ be the Banach space with the norm

$$\|y\|_Y = \max(\|u\|, \|v\|) \text{ for } y = (u, v) \in Y. \tag{3}$$

Let D be a subset of X. In the following, vectors and matrices mean n-dimensional vectors and $n \times n$-matrices, respectively. We assume that the given approximate solution $c(t)$ is an element of X. We now define an operator $F : D \subset X \to Y$ by

$$Fx = \left(\frac{\mathrm{d}x}{\mathrm{d}t} - f(x), g(x) \right). \tag{4}$$

Then we can rewrite the original problem as the following operator equation:

$$Fx = 0. \tag{5}$$

2.2. Existence Theorem of Solution for Eq.(5)

In this subsection, we shall introduce the existence theorem of the solution for Eq.(5).

In the following, we assume that $f : X \to X$ is continuously Fréchet differentiable with respect to x. The

Jacobi matrix of f with respect to x is denoted by $f_x(x)$, and the Fréchet derivative of g is denoted by l. Then it is easy to see that $F: X \to Y$ is Fréchet differentiable for an element x of D and the Fréchet derivative $F_x(x): D \to Y$ is given as follows:

$$F_x(x)h = \left(\frac{\mathrm{d}h}{\mathrm{d}t} - f_x(x)h, lh \right), \tag{6}$$

where $h \in D$.

For a real matrix valued step function $A(t)$ on J which approximate $f_x(x)$ and for the vector valued continuous linear functional $l = \sum_{k=1}^{n} B_k x(t_k)$, respectively, we define the following linear operator

$$Lh = \left(\frac{\mathrm{d}h}{\mathrm{d}t} - A(t)h, lh \right). \tag{7}$$

Let $\Phi(t)$ be the fundamental matrix of the linear homogeneous differential system

$$\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t} = A(t)\Phi(t) \tag{8}$$

satisfying $\Phi(0) = E$, where E is the unit matrix.

Lemma 2.1 Let $G = l[\Phi]$ be the matrix whose column vectors are $l[\phi_i(t)] (i = 1, 2, \cdots, n)$, where $\phi_i(t)$ are the column vectors of the matrix $\Phi(t)$.

If G is invertible, then L is also invertible and then, for $(u, v) \in Y$, $L^{-1}(u, v): Y \to X$ is described as

$$\begin{aligned} & L^{-1}(u, v) \\ &= \int_{-1}^{1} \Phi(t) \left(G^{-1} \left(\sum_{k=1}^{j} B_k \Phi(t_k) \right) - h(s-t)E \right) \cdot \\ & \Phi^{-1}(s)u(s)\mathrm{d}s + \Phi(t)G^{-1}v, \end{aligned} \tag{9}$$

where $h(s - t)$ is the Heviside's step function. □

Now we assume that G is invertible. We consider a Newton-like operator $k: X \to X$

$$k(x) = L^{-1}(f(x) - A(t)x, -b). \tag{10}$$

Let T be a closed convex subset of X and let be $T \ni c$. Let B be a closed ball of X whose midpoint is the origin and the radius is 1. We now introduce the following theorem.

Theorem 2.1 If

$$\{kx | x \in T\} \subset T \tag{11}$$

and

$$\max_{\substack{\hat{x} \in T \\ \check{x} \in B}} \|L^{-1}((f_x(\hat{x}, t) - A(t))\check{x}, 0)\| < 1 \tag{12}$$

hold, there is a fixed point x^* of k uniquely in T. □

This theorem is proved by Banach's contraction mapping theorem.

As the conventional method, T is calculated as

$$T = c + 2\|k(c) - c\|.$$

If we can calculate (11) and (12) by computers, we can check whether the solution exists or not by computers.

2.3. Affine Arithmetic[2]

In this subsection, we introduce Affine Arithmetic.

Definition 2.1 Let be $a_i \in \mathrm{R}$ for $i \in \{0, 1, \cdots, m\}$. Let be $-1 \leqq \varepsilon_i \leqq 1$ for $i \in \{1, \cdots, m\}$. The form

$$a_0 + \sum_{i=1}^{m} a_i \varepsilon_i$$

is called Affine Form and it describes the set

$$\{a_0 + \sum_{i=1}^{m} a_i \varepsilon_i | -1 \leqq \varepsilon_i \leqq 1, i \in \{0, 1, \cdots, m\}\}.$$

A set of Affine Forms which describe subsets of a space U is denoted by $\mathcal{A}(U)$. □

Definition 2.2 For $a^{(1)}, a^{(2)} \in \mathcal{A}(\mathrm{R})$, operations

$$\{a^{(1)} * a^{(2)} | * \in \{+, -, \times, /\}\}$$

and

$$\{\phi(a^{(1)}) | \phi \in \{\sin, \cos, \tan, \exp, \log, \cdots\}\}$$

are determined as

- the result is also an Affine Form
- the set described by the result holds

  $$a^{(1)} * a^{(2)} \supset \{x^{(1)} * x^{(2)} | x^{(1)} \in a^{(1)}, x^{(2)} \in a^{(2)}\}$$

  and

  $$\phi(a^{(1)}) \supset \{\phi(x^{(1)}) | x^{(1)} \in a^{(1)}\},$$

  respectively.

Let $a^{(1)}$ and $a^{(2)}$ be

$$a^{(1)} = a_0^{(1)} + \sum_{i=1}^{m} a_i^{(1)} \varepsilon_i \quad \text{and} \quad a^{(2)} = a_0^{(2)} + \sum_{i=2}^{m} a_i^{(2)} \varepsilon_i.$$

Addition and Subtraction between $a^{(1)}$ and $a^{(2)}$ are operated as

$$a^{(1)} \pm a^{(2)} = a_0^{(1)} \pm a_0^{(2)} + \sum_{i=1}^{m} (a_i^{(1)} \pm a_i^{(2)}) \varepsilon_i.$$

Addition and Subtraction between $a^{(1)}$ and a constant number $c \in \mathrm{R}$ are operated as

$$a^{(1)} \pm c = a_0^{(1)} \pm c + \sum_{i=1}^{m} a_i^{(1)} \varepsilon_i.$$

Multiplication between $a^{(1)}$ and $a^{(2)}$ is operated as

$$\begin{aligned} a^{(1)} \times a^{(2)} &= a_0^{(1)} a_0^{(2)} + \sum_{i=1}^{m} (a_0^{(1)} a_i^{(2)} + a_i^{(1)} a_0^{(2)}) \varepsilon_i \\ &+ (\sum_{i=1}^{m} |a_i^{(1)}|) \cdot (\sum_{i=1}^{m} |a_i^{(1)}|) \varepsilon_{m+1}, \end{aligned}$$

where $\varepsilon_{m+1} \in \mathrm{R}$ is a new symbol and satisfies $-1 \leqq \varepsilon \leqq 1$. Various unary operations for $a^{(1)}$, for example, the reciprocal of $a^{(1)}$, the square root of $a^{(1)}$, the sine function of $a^{(1)}$, and so on, have also been proposed but is not described here because of the sake of papers. □

2.4. Functional Arithmetic

For arithmetic on a functional space, we use some kinds of polynomials. Here we introduce Power polynomial

$$\sum_{i=1}^{l} \alpha_i t^i \quad (\alpha_i \in \mathrm{R})$$

and its arithmetic.

Definition 2.3 Let $c_1(t)$ and $c_2(t)$ be power polynomials described as below:

$$c_1(t) = \sum_{i=1}^{l} \alpha_i^{(1)} t^i \quad \text{and} \quad c_2(t) = \sum_{i=1}^{l} \alpha_i^{(2)} t^i.$$

Addition and subtraction between $c_1(t)$ and $c_2(t)$ are defined as

$$c_1(t) \pm c_2(t) = \sum_{i=1}^{l} (\alpha_i^{(1)} \pm \alpha_i^{(2)}) t^i.$$

Multiplication between $c_1(t)$ and $c_2(t)$ is defined as

$$c_1(t) \times c_2(t) = \sum_{i=0}^{2l} (\sum_{j=\max\{0,i-l\}}^{\min\{i,l\}} (\alpha_j^{(1)} \alpha_{\min\{i,l\}}^{(2)}) t^i).$$

The absolute value of $c_1(t)$ is overestimated as

$$|c_1| < \sum_{i=0}^{l} |\alpha_i^{(1)}|.$$

The differentiation of $c_1(t)$ with respect to t is calculated as

$$\frac{\mathrm{d}c_1}{\mathrm{d}t} = \sum_{i=0}^{l-1} (i+1) \alpha_{i+1}^{(1)} t^i. \tag{13}$$

The integration of $c_1(t)$ from -1 to t is calculated as

$$\int_{-1}^{t} c_1 \mathrm{d}t = \sum_{i=1}^{l+1} \frac{1}{i} \alpha_{i-1}^{(1)} t^i.$$

□

Let P be a set of power polynomials.

2.5. Functional Affine Arithmetic

In this section, we extend Affine Arithmetic to that on P.

Definition 2.4 Let be $a_i(t) \in P$ and let be $\varepsilon_i(t) \subset C[J]$ for $i \in \{1, \cdots, m\}$ and let be $a_0(t) \in P$. Let be $-1 \leqq \varepsilon_i(t) \leqq 1$ for $i \in \{0, 1, \cdots, m\}$. The form

$$a_0(t) + \sum_{i=1}^{m} a_i(t) \varepsilon_i(t)$$

is called Affine Form function and it describes a set of function $x(t)$ satisfying

$$x(t) \in a_0(t) + \sum_{i=1}^{m} a_i(t) \varepsilon_i$$

for all $t \in J$. □

Definition 2.5 For $a^{(1)}(t), a^{(2)}(t) \in \mathcal{A}(C[J])$, operations,

$$\{a^{(1)}(t) * a^{(2)}(t) | * \in \{+, -, \times, /\}\}$$

and

$$\{\phi(a^{(1)}(t)) | \phi \in \{\sin, \cos, \tan, \exp, \log, \cdots\}\},$$

are determined as

- the result is also an Affine Form function
- the set described by the result holds

$$a^{(1)}(t) * a^{(2)}(t) \supset \{x^{(1)}(t) * x^{(2)}(t) | x^{(1)}(t) \in a^{(1)}(t), x^{(2)}(t) \in a^{(2)}(t)\}$$

  and

$$\phi(a^{(1)}(t)) \supset \{\phi(x^{(1)}(t)) | x^{(1)}(t) \in a^{(1)}(t)\},$$

  respectively.

Let $a^{(1)}(t)$ and $a^{(2)}(t)$ be

$$a^{(i)}(t) = a_0^{(1)}(t) + \sum_{i=1}^{m} a_i^{(i)}(t) \varepsilon_i(t) \quad (i \in \{1, 2\}).$$

Addition and Subtraction between $a^{(1)}(t)$ and $a^{(2)}(t)$ are operated as

$$a^{(1)}(t) \pm a^{(2)}(t) = a_0^{(1)}(t) \pm a_0^{(2)}(t) + \sum_{i=1}^{m} (a_i^{(1)}(t) \pm a_i^{(2)}(t)) \varepsilon_i$$

Addition and Subtraction between $a^{(1)}(t)$ and a power polynomial $c(t) \in P$ are operated as

$$a^{(1)}(t) \pm c(t) = a_0^{(1)}(t) \pm c(t) + \sum_{i=1}^{m} a_i^{(1)}(t) \varepsilon_i(t).$$

Multiplication between $a^{(1)}(t)$ and $a^{(2)}(t)$ is operated as

$$\begin{aligned} a^{(1)}(t) \times a^{(2)}(t) &= a_0^{(1)}(t) a_0^{(2)}(t) \\ &+ \sum_{i=1}^{m} (a_0^{(1)}(t) a_i^{(2)}(t) + a_i^{(1)}(t) a_0^{(2)}(t)) \varepsilon_i(t) \\ &+ (\sum_{i=1}^{m} |a_i^{(1)}(t)|) \cdot (\sum_{i=1}^{m} |a_i^{(2)}(t)|) \varepsilon_{m+1}(t), \end{aligned}$$

where $\varepsilon_{m+1}(t)$ is a new symbol and satisfies $-1 \leqq \varepsilon_{m+1}(t) \leqq 1$. The absolute value of $a^{(1)}(t)$ is overestimated as

$$|a^{(1)}(t)| < \sum_{i=0}^{m} |a_i^{(1)}(t)|.$$

The integration of $a^{(1)}(t)$ from -1 to t is overestimated as

$$\int_{-1}^{t} a^{(1)}(\tau) \mathrm{d}\tau = \int_{-1}^{t} a_0^{(1)}(\tau) \mathrm{d}\tau + \sum_{i=m+1}^{2m} \int_{p}^{t} |a_i^{(1)}(\tau)| \mathrm{d}\tau \varepsilon_i(t),$$

where $\varepsilon_i(t) \in C[J]$s ($i \in \{m+1, m+2, \cdots, 2m\}$) is the new noise symbols and satisfies $-1 \leqq \varepsilon_i \leqq 1$, respectively. □

2.6. Inclusion of $h(s-t)$

When we exactly treat $h(s-t)$ as

$$h(s-t) = \begin{cases} 0 & (s < t), \\ 1 & (s > t), \end{cases}$$

we cannot calculate (9) with high accuracy because $u(s)$ is Interval Function. In this subsection, we consider an inclusion $H(s-t) = [\bar{h}(t,s), \underline{h}(t,s)]$ of $h(s-t)$ in order to avoid the above difficulty, where both $c_1(t,s)$ and $c_2(t,s)$ are polynomials.

A function

$$p(s,a) = \frac{\exp(2as)}{\exp(2as)+1}$$

is a good approximation of $h(s)$ for large a. Using this function, we have

$$\begin{aligned} p(s - \frac{1}{2}\log(\frac{1-b}{b}), a) - b &\leqq h(s) \\ &\leqq p(s + \frac{1}{2}\log(\frac{1-b}{b}), a) + b. \end{aligned}$$

Using Taylor expansion, we can obtain an upper bound $c_1(s)$ of $p(s+\frac{1}{2}\log(\frac{1-b}{b}), a)+b$ and an lower bound $c_2(s)$ of $p(s-\frac{1}{2}\log(\frac{1-b}{b}), a)-b$. Since $c_1(s)$ is also an upper bound of $h(s)$ and $c_2(s)$ is also an lower bound of $h(s)$, we can set

$$H(s-t) = [c_1(s-t), c_2(s-t)].$$

3. Numerical Verification

In order to obtain T, we have to calculate $k(c)$. Since $c(t)$ and $A(t)$ are given as a step function, we can calculate $f(c) - A(t)c$ using piecewise machine Interval Arithmetic and we can obtain the inclusion of $f(c) - A(t)c$ as an Interval Function whose endpoints functions are step functions. Using some methods[5],[3], we can obtain the inclusions of $\Phi(t)$ and $\Phi^{-1}(t)$ as Interval Functions whose endpoints functions are polynomials. Since $h(s-t)$ is obtained as an Interval Function whose endpoints functions are polynomials, we can obtain $k(c)$ as an Interval Function whose endpoints functions are polynomials. After evaluating the norm $\|k(c) - c\|$, we can obtain T as an Interval Function whose endpoints functions are step functions.

In order to confirm that conditions (11) and (12) hold, we have to calculate the left-hands of these conditions. After translating Interval Functions T and B into Affine Form, $\{f(x) - A(t)x | x \in T\}$ and $\{(f_x(\hat{x}) - A(t))\check{x} | \hat{x} \in T, \check{x} \in B\}$ can be calculated using piecewise Affine Arithmetic. Since $\Phi(t)$, $\Phi^{-1}(t)$ and $H(t-s)$ are obtained as Interval Function and they can be translated into Affine Forms, We can obtain left-hands of conditions (11) and (12) using Functional Affine Arithmetic.

4. Numerical Example

Let us consider the equation described as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 - (x_1 - t)^3 + t + 0.1 \end{pmatrix}, \tag{14}$$

$$g(x) = \begin{pmatrix} x_1(0) - 0.1 \\ x_1(1) - 1.1 \end{pmatrix}. \tag{15}$$

We took

$$c(t) = \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} t + 0.1 \\ 1 \end{pmatrix}$$

as an approximate solution of (14), (15).

We set T as $T = c + 2\|k(c) - c\|$. By translating T into an Affine Form described as

$$T = (c_1(t) + b\varepsilon_{c_1}(t), c_2(t) + b\varepsilon_{c_2}(t))^{\mathrm{tr}}.$$

and by calculating an inclusion of $\{kx | x \in T\}$ and an upper bound of

$$\max_{\substack{\hat{x}\in T \\ \check{x}\in T}} \{\|L^{-1}(f_x(\hat{x}(t), t) - A(t)), l\hat{x} - g(\hat{x}))\check{x}(t)\|\}$$

using Affine Arithmetic, we have the sufficient condition for Theorem 2.1 and can find a unique solution for Eq.(14), (15) in T.

Acknowledgment

The authors would like to thank Prof. Masahide Kashiwagi of Waseda University for their valuable discussions.

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