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Experiments on Lorenz system

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Abstract– Experiments using a simple circuit described by the Lorenz equations are presented. The circuit consists of two analog multipliers and seven passive elements, is robust and very easy to build. It is intended to allow laboratory experiments giving support to Lorenz chaos theory and its applications. By varying the circuit parameters a large class of phenomena can be observed, such as stable and chaotic symmetrical and non-symmetrical orbits, bifurcations, noisy periodicity, hysteresis, transient chaos and intermittency. Numerical experiments using an approximate function to observed Lorenz maps are also described.

1. Introduction

In the last decades the Lorenz system [1] has been intensely studied and used as a paradigm in chaos research and applications. However, due to practical difficulties, few experimental works have been published which give support to the theory. Aiming at contributing in this direction, this work reports on a new circuit realization of the Lorenz system and its use in laboratory experiments. Existing Lorenz circuits are in the analog computer form made of a large number of components [2]. In the present approach fewer components are employed. The proposed circuit is exactly described by the Lorenz equations, is easy to build and is robust in the sense that the chaotic attractor persists under perturbation of parameters. Other circuits showing Lorenz-like butterfly attractors have been reported in [3] and [4]. However, as explained in those works, their circuit equations are not topologically equivalent to the Lorenz equations. To achieve fully equivalent systems it is essential the use of multiplier circuits, because the nonlinearities involve product terms of variables. The proposed new circuit comprises two analog multipliers, four resistors, two capacitors and one inductor, possibly the simplest Lorenz circuit. Obvious limitations are caused mainly by non-idealities in the analog multipliers, such as low slew rate and limited dynamic range. Accordingly, to reduce distortions, the circuit needs to be operated within appropriate intervals of parameter values. Though the parameter space may be restricted, most of the important aspects of Lorenz system can be observed. In the following sections the proposed circuit is described and some experimental results are presented. A numerical approach for studying the experimentally observed Lorenz maps is also presented.

2. Circuit model

The proposed circuit is shown in Fig. 1(a), where two analog multipliers are employed. The first multiplier is connected as a voltage-controlled voltage source. The other multiplier is connected, via R_4, as a voltage-controlled current source. The transfer function of each multiplier is given by $W=0.1(X_1-X_2)(Y_1-Y_2)+Z$, therefore the output voltage of the first multiplier is $W=-0.1V_1V_2+V_1$ and the output current of the second multiplier is $I_{R4}=0.1IV_1R_2/R_4$. The equivalent circuit is shown in Fig. 1(b). The circuit equations are:

$$
\begin{aligned}
C_1\frac{dV_1}{dt} &= I-\frac{V_1}{R_1}\\
L\frac{dI}{dt} &= -\frac{V_1V_2}{10}-R_2I\\
C_2\frac{dV_2}{dt} &= \frac{R_2IV_1}{10R_4}-\frac{V_2+E}{R_3}
\end{aligned}
\qquad (1)
$$

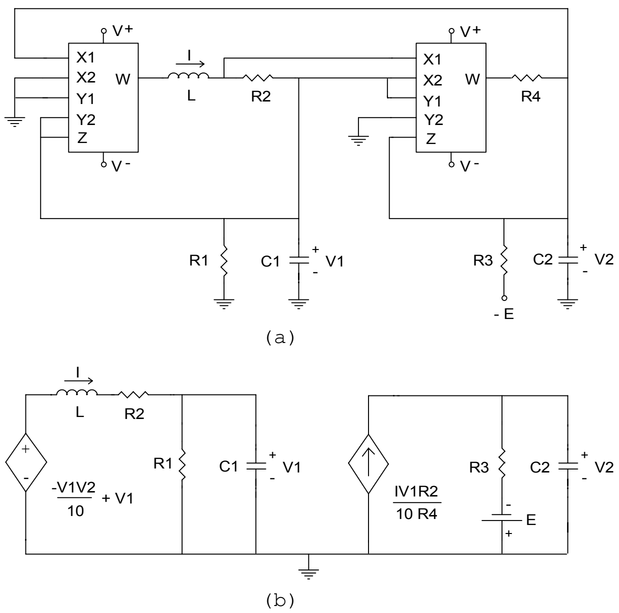


Fig. 1. (a) The proposed Lorenz circuit and (b) its equivalent circuit.

Rescaling time by $t \rightarrow \tau\sigma R_1 C_1$ and defining the following dimensionless quantities:

$$\sigma = \frac{L}{R_1 R_2 C_1} \quad , \quad b = \frac{L}{R_2 R_3 C_2} \quad , \quad r = \frac{E R_1}{10 R_2}$$
$$x = \frac{V_1}{10}\sqrt{\frac{L}{R_2 R_4 C_2}} \quad y = \frac{R_1 I}{10}\sqrt{\frac{L}{R_2 R_4 C_2}} \quad z = \frac{V_2 R_1}{10 R_2} + r \tag{2}$$

then, by combining (1) and (2), Lorenz equations [1] are obtained:

$$\begin{aligned}\frac{dx}{d\tau} &= \sigma(y - x)\\ \frac{dy}{d\tau} &= x(r - z) - y\\ \frac{dz}{d\tau} &= xy - bz\end{aligned} \tag{3}$$

Note that the parameters σ, b and r can be independently varied by C_1, R_3 and E, respectively.

3. Experiments

3.1. Phenomena visualization

An experimental circuit was constructed using two AD633 multipliers powered from ±15V supplies and the following fixed elements: L=10mH, R_1=1.5kΩ and R_4=1kΩ. By varying the circuit parameters a large class of phenomena can be observed, such as stable and chaotic symmetrical and nonsymmetrical orbits, bifurcations, noisy periodicity, hysteresis, extra-twisting, transient chaos and intermittency. By Eqs. (2), the popular values σ=10 and b=8/3 [1] can be obtained with R_2=50Ω, R_3=200kΩ, C_1=13.3nF, and C_2=330pF, but in that case the bifurcation parameter r, in the chaotic region without waveform clipping, ranges from only r~25 to r~35. Reducing b, or both b and σ, extends the r range. For example, with R_2=100Ω, R_3=900kΩ, C_1=23.6nF and C_2=1nF (corresponding to σ=2.82 and b=0.11) r can be varied from r~5 to r~76.

As examples of observable phenomena, some results are now presented. Fig. 2 shows a typical V_1-V_2 trajectory. By decreasing E while keeping fixed all the other circuit parameters, there is a point where the chaotic attractor suddenly disappears due to an inverted bifurcation [5]. Below this point transient behavior can be observed by periodically discharging the capacitor C_1. Using this method, it is possible to detect and examine the region where occurs the first homoclinic orbit, as explained in [1] (Fig. 3). In Fig. 4(a) a Lorenz map (a plot of successive local maxima of V_2) is shown, corresponding to the attractor in Fig. 2. Figure 4(b) shows a double-cusp Lorenz map (corresponding to extra-twisting [1] around the V_2-axis). One-dimensional bifurcation diagrams can be obtained by displaying the peak values of $V_2(t)$ as a function of a varying parameter. They can be used for studying hysteresis for example, which can be observed by first varying the chosen parameter in one direction and then in the other, as illustrated in Fig. 5 for varying E.

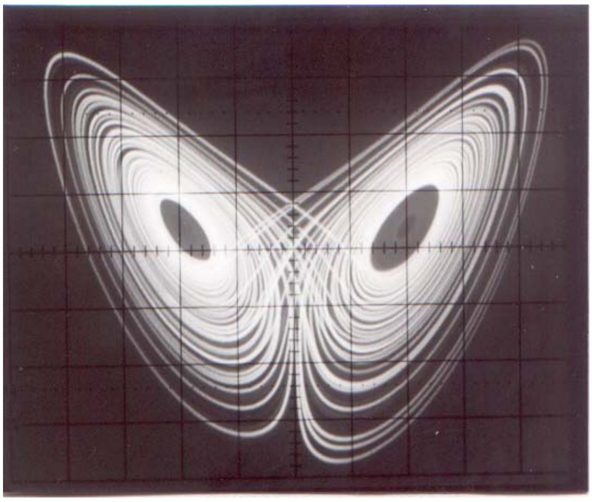


Fig. 2. Observed trajectory using E=15V, R_2=100Ω, R_3=160kΩ, C_1=22nF and C_2=1nF. The chaotic attractor disappears at E=13.8V. Horizontal: V_1 (2V/div). Vertical: V_2 (2V/div).

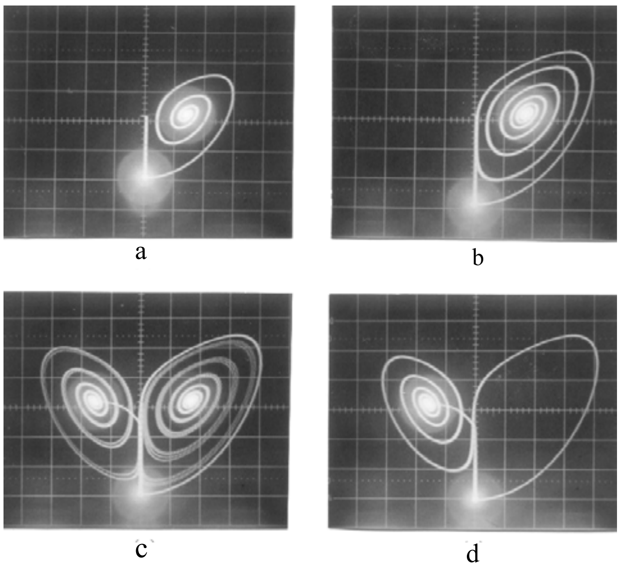
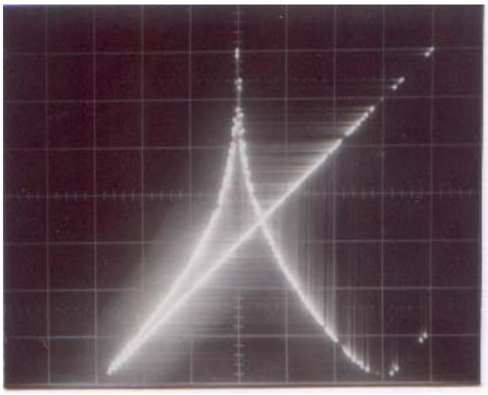
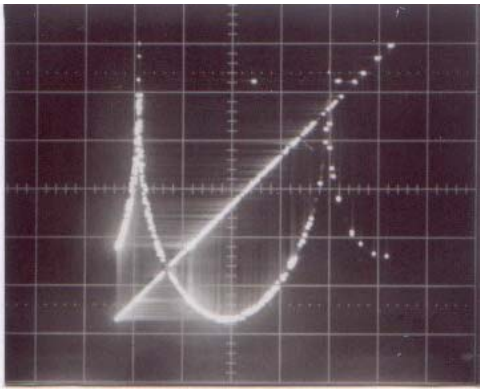


Fig. 3. Observed trajectories around the first homoclinic orbit (a) E=2.77V; (b) E =3.62V; (c) E =3.69V; (d) E=3.75V. The other circuit parameters are the same as in Fig. 2. Between (b) and (d) an unstable behavior (c) is observed. Horizontal: V_1 (1V/div). Vertical: V_2 (1V/div).



(a)



(b)

Fig. 4. (a) Experimental Lorenz map for the attractor shown in Fig. 2. (b) A double-cusp Lorenz map (R_2=100Ω, R_3=460kΩ, C_1=14.7nF, C_2=500pF and E=23V). Horizontal: $V_{2(\mathrm{peak})}$ (1V/div). Vertical: $V_{2(\mathrm{peak})}$ (1V/div).

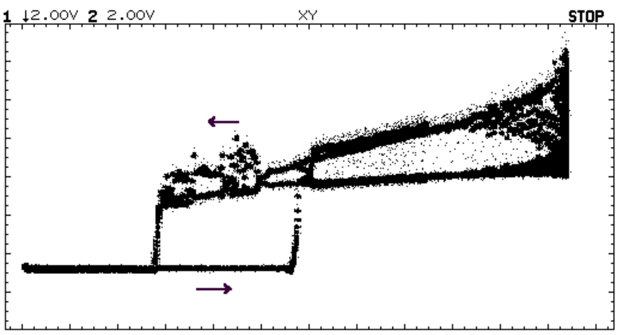


Fig. 5. Bifurcation diagram showing hysteresis observed by varying E in the directions indicated by the arrows (R_2=100Ω, R_3=406kΩ, C_1=14.7nF, C_2=1nF). Horizontal: E (2V/div). Vertical: $V_{2(\mathrm{peak})}$ (2V/div).

3.2. Numerical approach

By using an approximate curve to experimental points recorded from the observed Lorenz map it is possible to investigate some phenomena via one-dimensional iterated dynamics. Let $u(n)$ denote the n[th] peak value of $V_2(t)$. Then $u(n+1)\approx U(u(n))$, where

$$\begin{aligned} U(u) &= U_0 - a_1(u-u_0)^{\beta}[1+a_2(u-u_0)+ \\ &\quad + a_3(u-u_0)^2\,, \quad u \geq u_0 \\ U(u) &= U_0 - a_4(u_0-u)^{\beta}\left[1+a_5(u_0-u)\right], \quad u < u_0 \end{aligned} \tag{4}$$

Function (4) is a here proposed modified version from that used in [5]. β can be calculated by the theoretical formula [5]:

$$\beta = \frac{2b}{\left|\sigma+1-\sqrt{(\sigma+1)^2+4\sigma(r-1)}\right|} \tag{5}$$

Given the coordinates (u_0,U_0) of the cusp maximum, estimated from the experiment, the other parameters can be determined by the least squares technique or other method. Fig 6 shows an example of experimental Lorenz map and its fitting curve. Lorenz map properties can be investigated by writing Eq. (4) as a function of a varying circuit parameter. This is illustrated in Fig. 7, which shows results of numerical experiments on Lyapunov exponents, hysteresis, transient chaos and intermittency. The used curve parameters are presented in the Appendix.

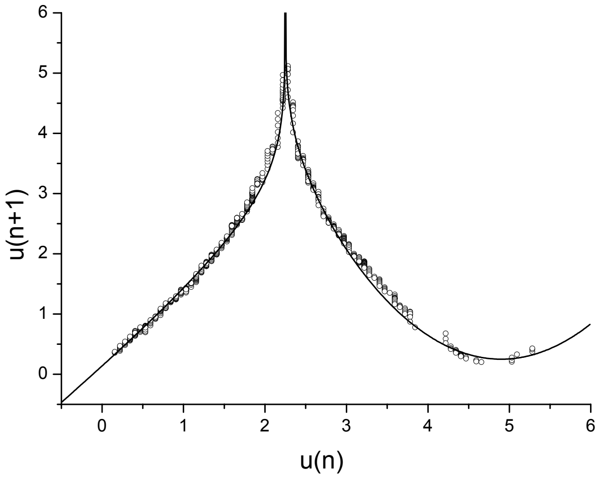


Fig. 6. Experimental Lorenz map (points) for R_2=100Ω, R_3=190kΩ, C_1=22nF, C_2=1nF and E=15V, and its fitting function (a_1=7.0V$^{1-\beta}$, a_2=0.254V^{-1}, a_3=-0.055V^{-2}, a_4=7.4V$^{1-\beta}$, a_5=0.109 V^{-1}, u_0=2.25V, U_0=10.0V, β=0.084).

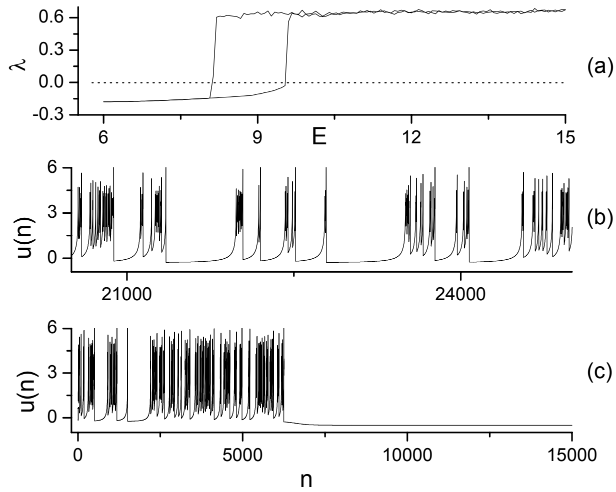


Fig. 7. Numerical experiments on Lorenz map using the parameters presented in the Appendix. a) Lyapunov exponent λ for varying E, showing hysteresis. b) Intermittent chaos (E=12.273V). c) Transient chaos (E=12.270V).

4. Conclusion

Laboratory experiments using a simple circuit realization of the Lorenz system are presented. Numerical experiments using an approximate function to Lorenz maps displayed by the circuit variable V_2 are also described. An interesting feature of the circuit is that the parameter r can be varied linearly with a bias voltage (therefore modulation of r by an external signal can easily be accomplished). Limitations typical of electronic circuits reduce the permitted range of a bifurcation parameter. To partially overcome this problem and thus explore the richness of the Lorenz system dynamics, an adopted solution was to reduce the size of the attractor by working with smaller b and σ values than the popular b=8/3 and σ=10. Such a restriction is not disadvantageous because the system qualitative behavior is essentially preserved.

As a final comment, it is worth noting that by introducing slight modifications in the circuit of Fig. 1 a simple circuit realization of the Chen system [6] can be obtained ─ this scheme will be described elsewhere.

Appendix

The hysteresis curve of Fig. 7(a) is calculated using the following approximate relations derived from experimental data (σ=3, b=0.526, 9<r<40, where r=1.5E):

$$
\begin{aligned}
a_1 &= 2.9 + 0.3E \\
a_2 &= .23 - 2.18\exp(-\frac{E}{3.67}) \\
a_3 &= -0.04 \\
a_4 &= 2.19 + 0.36E \\
a_5 &= 0.07 + 0.002E \\
u_o &= 0.83 + 0.09E \\
U_o &= 1.23 + 0.59E
\end{aligned}
\tag{A1}
$$

To find the intermittent chaos and transient chaos waveforms of Fig. 7(b) and Fig. 7(c) the following relations are used (σ=3, b=0.97 , 15<r<20, where r=1.5E):

$$
\begin{aligned}
a_1 &= 4.334 + 0.033E \\
a_2 &= 0.105 + 0.021E \\
a_3 &= -0.0793 + 0.001E \\
a_4 &= 5.171 + 0.004E \\
a_5 &= -0.1324 + 0.018E \\
u_o &= 1.757 + 0.123E \\
U_o &= 2.22 + 0.495E
\end{aligned}
\tag{A2}
$$

Acknowledgments

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