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# Dissipative Solitons in the Discrete Complex Ginzburg-Landau Model

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**Abstract**—We study, analytically, the discrete complex cubic-quintic Ginzburg-Landau (dCCQGL) equation. We find a set of exact solutions for this equation which includes as particular cases periodic solutions in terms of elliptic Jacobi functions, bright and dark soliton solutions, and constant magnitude solutions. We perform the numerical simulations for exact solitary waves. We discuss the relation of these solutions to solutions of the continuous complex Ginzburg-Landau model.

## 1. Introduction

The complex Ginzburg-Landau (CGL) equation plays an important role in various branches of science. Dissipative solitons of the CGL equation, studied by many authors, are possible due to the interplay between linear and nonlinear gain, nonlinearity, dispersion, and dissipation. In this context, "soliton" means a localized wave structure (i.e., they do not necessarily interact elastically).

Discrete solitons in nonlinear lattices are possible in various areas, e.g. biology, atomic chains, solid state physics, electrical lattices and Bose-Einstein condensates [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. Discrete solitons also exist in arrays of coupled nonlinear optical waveguides [11]. Photonic crystals can support "discrete solitons" with interesting properties.

In the discrete nonlinear Schrödinger (dNLS) equation, the term $\psi_{n+1} - 2\psi_n + \psi_{n-1}$ plainly approximates a second derivative term for a continuous system and so physically represents diffraction. A transform eliminates the term $-2\psi_n$, thus indicating that what is occurring is nearest-neighbour coupling. Hence, a realistic discrete system features diffraction-type effects.

The dNLS equation has been used [12] to model the propagation of discrete self-trapped beams in an array of weakly-coupled nonlinear optical waveguides, but it is not completely integrable. The integrable discrete nonlinear Schrödinger equation (Ablowitz-Ladik (AL) system) is

$$i\frac{d\psi_n}{dt} + \frac{D}{2}(\psi_{n+1} - 2\psi_n + \psi_{n-1}) + |\psi_n|^2(\psi_{n+1} + \psi_{n-1}) = 0\,. \quad (1)$$

The integrability of some 'deformed' dNLS equations was investigated in [13].

A discrete analogue of the complex cubic-quintic Ginzburg-Landau equation was studied in [14]. Using a perturbation technique, they found a soliton solution which is valid at small values of the dissipative terms for this equation. A different discrete complex cubic-quintic Ginzburg-Landau equation was studied in [15].

In [16], we studied the discrete complex Ginzburg-Landau equation (dCGLE)

$$i\frac{d\psi_n}{dt} + \left(\frac{D}{2} - i\beta\right)(\psi_{n+1} - 2\psi_n + \psi_{n-1}) + (1 - i\epsilon)|\psi_n|^2(\psi_{n+1} + \psi_{n-1}) = i\delta\psi_n\,. \quad (2)$$

Several exact solutions were derived in this case, but they were unstable in the numerical study [17], reflecting the situation in the complex cubic continuous Ginzburg-Landau equation.

In this paper, we also study a discrete equation, similar to that in [14, 15], but include a quintic nonlinearity which is non-local. We derive exact solutions which are valid at arbitrary values of dissipative terms. So we are not limited to small values. Specifically, we consider a model of a dissipative system, *viz.* the following discrete complex cubic-quintic Ginzburg-Landau equation (dCCQGLE)

$$i\frac{d\psi_n}{dt} + \left(\frac{D}{2} - i\beta\right)(\psi_{n+1} - 2\psi_n + \psi_{n-1}) + (1 - i\epsilon)\,\psi_n\,|\psi_n|^2 + (\nu - i\,\mu)\,|\psi_n|^4\,(\psi_{n+1} + \psi_{n-1}) - i\delta\,\psi_n = 0\,. \quad (3)$$

The continuous limit of (3) is the complex cubic-quintic Ginzburg-Landau equation (CCQGLE) [18]

$$i\frac{d\psi}{dt} + \left(\frac{D}{2} - i\beta\right)\psi_{xx} + (1 - i\epsilon)|\psi|^2\psi + (\nu - i\,\mu)|\psi|^4\psi = i\delta\psi\,. \quad (4)$$

which has many applications in describing non-equilibrium systems, phase transitions, and wave propagation phenomena. In the limit of $\beta = \epsilon = \nu = \mu = \delta = 0$, eq.(3) is reduced to the (non-integrable) discrete nonlinear Schrödinger equation

$$i\frac{d\psi_n}{dt} + \frac{D}{2}(\psi_{n+1} - 2\psi_n + \psi_{n-1}) + \psi_n\,|\psi_n|^2 = 0\,. \quad (5)$$

If we take the continuous limit of eq.(3) with $\psi_n = a\Psi_n$, $\tau = a^2 t$, $x = na$ and $\delta = \delta_a a^2$ ($a$ is a small lattice parameter), we have the complex quintic Swift-Hohenberg type equation

$$i\Psi_\tau + \left(\frac{D}{2} - i\beta\right)\Psi_{xx} + (1 - i\epsilon)|\Psi|^2\Psi$$
$$+2a^2(\nu_a - i\mu_a)|\Psi|^4\Psi_n + \frac{a^2}{12}\left(\frac{D}{2} - i\beta\right)\Psi_{xxxx} = i\delta_a\Psi\,, \quad (6)$$

which also has many applications in describing non-equilibrium systems.

## 2. Exact soliton solutions

Stationary solutions of eq.(3) are defined by

$$\psi_n = \phi_n\, e^{-i\omega t}\,. \quad (7)$$

The Hirota method can be applied to obtain selected exact solutions of eq.(3) [19]. In order to do this we substitute $\psi_n(t) = \phi_n(t)e^{-i\omega t} = \frac{g_n(t)}{f_n(t)}e^{-i\omega t}$, $\psi_n^*(t) = \phi_n^*(t)e^{i\omega t} = \frac{g_n^*(t)}{f_n(t)}e^{i\omega t}$ with real $f_n$, into eq.(3). As a result, we obtain the multilinear form. Then, the standard procedure of the Hirota method can be used to obtain the exact solutions listed in the following sections.

Solutions can be obtained only for certain relations between the coefficients of the equations. Namely, we set

$$\delta = \epsilon\omega\,, \quad \beta = \epsilon D/2\,, \quad \mu = \epsilon\nu\,, \quad (8)$$

and note that in this case the system simplifies to

$$\omega\phi_n + \frac{D}{2}(\phi_{n+1} - 2\phi_n + \phi_{n-1}) + \phi_n^3 + (\phi_{n+1} + \phi_{n-1})\,\nu\,\phi_n^4 = 0. \quad (9)$$

**Simple (constant) solution.** We take $D, \nu$ and $\epsilon$ arbitrary. Direct substitution shows that any constant $a$ is a solution, so long as $\omega = -a^2(1 + 2a^2\nu)$.

**Alternating constant solution.** Furthermore, $(-1)^n a$ is a solution for any constant $a$, so long as $\omega = 2D + 2a^4\nu - a^2$.

### 2.1. Bright soliton.

By using,for example, the Hirota method, we can find the explicit solution for the fundamental soliton with a constant phase across its profile.

We can write the two relevant solutions separately using the function sech. For the bright pulse solution, we need $D > 0$ but arbitrary, and $\nu < 0$ but arbitrary.

For convenience, we define $k$ using

$$\mathrm{sech}(k) = \sqrt{-2\nu D}\ (< 1).$$

The solution is then:

$$\phi_n = \sqrt{D\cosh(k)}\,\sinh(k)\,\mathrm{sech}(kn + \alpha), \quad (10)$$

with $\alpha$ arbitrary and $\omega = -2D\sinh^2(\frac{k}{2})$. This can also be expressed as

$$\phi_n = \frac{\left(p - p^{-1}\right)\sqrt{\frac{D(p+p^{-1})}{2}}}{p^{n+n_a} + p^{-n-n_a}}\,. \quad (11)$$

where $k = \log p$ and $n_a = \alpha/k$.

The soliton profile is shown in Fig.1a. The numerical simulations based on the original equation (3) show that this solution is stable. The results of the simulation are shown in Fig.1b. Small perturbations do not destroy the solution and tend to disappear as the soliton evolves in time. We recall that solitons of the dCGLE are unstable [17]. This shows that quintic terms are important in making the soliton stable.

Let us consider particular examples. For $D = 1$ and $\nu = -1/4$, we have the all-positive solution:

$$\phi_n = 2^{\frac{1}{4}}\,\mathrm{sech}\left(n\,\mathrm{arcsech}\left(\frac{1}{\sqrt{2}}\right)\right) \approx 1.189207\mathrm{sech}(0.8813736n)$$

This can also be expressed as

$$\phi_n = \frac{2.37841}{q^n + q^{-n}},\ q = 0.414214, \quad (12)$$

and is clearly positive everywhere.

### 2.2. Bright soliton with alternating sign

In the case of the bright alternating sign (spiked) soliton solution , we need $D < 0$ but arbitrary, and $\nu > 0$ but arbitrary, As before, we define $k$ through $\mathrm{sech}(k) = \sqrt{-2\nu D}(< 1)$. The solution is:

$$\phi_n = (-1)^n\sqrt{-D\cosh(k)}\,\sinh(k)\,\mathrm{sech}(kn + \beta), \quad (13)$$

with $\beta$ arbitrary and $\omega = +2D\cosh^2(\frac{k}{2})$.

For $D = -1$ and $\nu = 1/4$, we have the alternating sign solution:
$\phi_n = 1.189207\,(-1)^n\,\mathrm{sech}(0.8813736n)$, which also be expressed as

$$\phi_n = \frac{2.37841}{q^n + q^{-n}},\ q = -0.414214. \quad (14)$$

It is clearly positive for $n$ even and negative for $n$ odd.

### 2.3. Dark soliton.

In analogy with the previous case we can find the solution for the dark solitons. We can also introduce the parameter $k$ through $\cosh^4 k = -2D\nu(> 1)$. For $D < 0$ but arbitrary and $\nu > 0$ but arbitrary , such that $2D\nu < -1$, the plain dark soliton solution can be written in terms of hyperbolic functions:

$$\phi_n = \sqrt{-D}\,\mathrm{sech}(k)\,\tanh(k)\,\tanh(kn + \alpha_2) \quad (15)$$

where $\alpha_2$ is arbitrary and the frequency is $\omega = D\tanh^2 k$. This can also be expressed as

$$\phi_n = \frac{2\sqrt{-D}(p - p^{-1})}{(p + p^{-1})^2}\,\frac{p^{n+n_a} - p^{-n-n_a}}{p^{n+n_a} + p^{-n-n_a}}\,, \quad (16)$$

where $n_a = \alpha/k$.

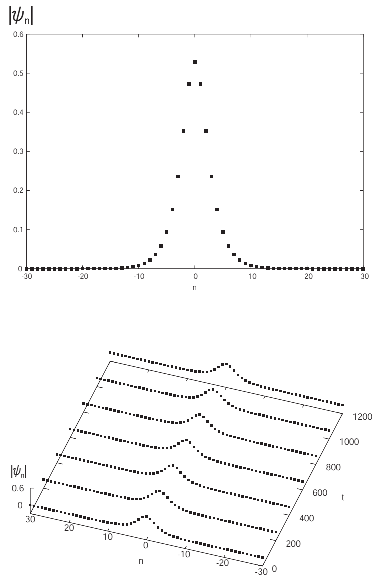


Figure 1: (a) Bright soliton profile for the equation parameters $D = 1$, $\nu = -0.4$. This implies that $k = 0.481212$ and the soliton is given by $\phi_n = 0.528686\,\mathrm{sech}[0.481212\,n]$. (b) Numerical simulation showing stationary soliton evolution in time for $\epsilon = 0.05$.

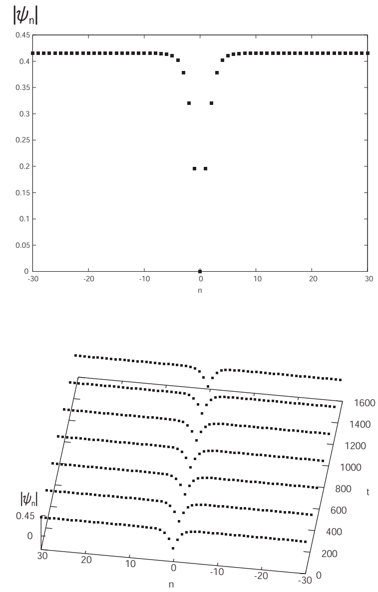


Figure 2: Dark soliton profile for the equation parameters $D = -1$, $\nu = 0.824898765$. Numerical simulation showing stable stationary dark soliton evolution in time for $\epsilon = 0.001$.

For example, $D = -1$, $\nu = 2$, we have $\phi_n = \frac{1}{2}\tanh(0.88137\,n)$.

The dark soliton profile is shown in Fig.2a. Numerical simulations showing the evolution of this solution in time are presented in Fig.2b. The solution is stable and evolves in time without changing. Moreover, small perturbations do not grow but rather disappear exponentially with time.

### 2.4. Alternating sign dark soliton.

We now describe the solution for the dark soliton with alternating sign values of $\phi_n$. We again define the parameter $k$ through $\cosh^4 k = -2\,D\,\nu\,(> 1)$. For $D > 0$ but arbitrary and with $\nu < 0$ but arbitrary , such that $2\,D\,\nu < -1$, as before, we can express this solution in terms of hyperbolic functions. The alternating sign dark solution can be written as

$$\phi_n = (-1)^n\,\sqrt{D}\,\mathrm{sech}(k)\,\tanh(k)\,\tanh(k\,n + \beta_2) \qquad (17)$$

where $\beta_2$ is arbitrary and the frequency $\omega$ is $\omega = D(1 + \mathrm{sech}^2 k)$. This can be expressed as

$$\phi_n = (-1)^n\,\frac{2\sqrt{D}(p - p^{-1})}{(p + p^{-1})^2}\,\frac{p^{n+n_a} - p^{-n-n_a}}{p^{n+n_a} + p^{-n-n_b}}, \qquad (18)$$

where $n_b$ is arbitrary.

For example, $D = +1$, $\nu = -2$, we have $\phi_n = (-1)^n\,\frac{1}{2}\tanh(0.88137\,n)$.

### 2.5. Periodic solutions

There are many periodic solutions with $\nu$ arbitrary, apart from sign, and $D$ arbitrary. Here are some examples:

$$\phi_n = \sqrt{\frac{-2}{5\,\nu}}\cos(n\,\pi/3),\ \textit{with}\ \ \omega = \frac{D}{2} + \frac{4}{25\,\nu}, \qquad (19)$$

$$\phi_n = \sqrt{\frac{-\sqrt{2}}{3\,\nu}}\cos(n\,\pi/4),\ \textit{with}\ \ \omega = D - \frac{D}{\sqrt{2}} + \frac{\sqrt{2}}{9\,\nu}, \qquad (20)$$

Eqs. (19) and (20 ) plainly require $\nu < 0$, while the following one needs $\nu > 0$.

$$\phi_n = \sqrt{\frac{2}{3\,\nu}}\cos(\frac{\pi}{6}(5 - 4\,n)),\ \textit{with}\ \ \omega = \frac{6D\,\nu - 1}{4\,\nu}. \qquad (21)$$

### 2.6. Elliptic function solutions

We now show that elliptic function solutions of the discrete equation also exist. The **Jacobi cn** function solution has the form

$$\phi_n = a\,\mathrm{cn}\left(\frac{n}{2}\,K(m),\,m\right), \qquad (22)$$

where $K$ is the complete elliptic integral of the first kind. We write $m = 1 - \sinh^4 b$, where $b$ is arbitrary. Then the amplitude is found from

$$a^2 = \frac{-2 \pm \sqrt{r}}{2\,\nu\,(3 + \cosh[2b])}\coth(b)\ \ (> 0), \qquad (23)$$

where $r = 4 + 17Dv - Dv\cosh(4b)$ $(> 0)$, and $\omega = D - a^2 - (D + 2a^2 v)\tanh(b)$. The requirement on $r$ means that we need to have either $Dv > 0$ or $Dv < -\frac{1}{4}$. This produces a sequence of period 8, with the $\phi_n$, $(n = 0, 1, \cdots)$ being given by $(a, a\tanh(b), 0, -a\tanh(b), -a, -a\tanh(b), 0, a\tanh(b), a, \cdots)$ with $b$ arbitrary. We now consider the special case of eq. (23 ) when $m = 0$. While the plus sign case in eq. (23) gives the zero solution, the minus sign gives $\phi = a\cos(n\pi/4)$, with $a^2 = -\frac{\sqrt{2}}{3v}$ with $\omega$ given in eq. (20) above. In fact, it is clearly the same solution as eq. (20). There are also Jacobi **sn** function solutions.

## 3. Conclusions

We have studied, analytically, the discrete complex cubic-quintic Ginzburg-Landau equation. We have found a set of exact solutions which includes, as particular cases, bright and dark soliton solutions, constant magnitude solutions, periodic solutions in terms of elliptic Jacobi functions in general form, and particular cases of periodic solutions. We have given the range of parameters where various of these exact solutions exist. Using numerical simulations, we have found that (some) soliton solutions of the discrete complex cubic-quintic Ginzburg-Landau equation are stable, in contrast to the soliton solutions of the discrete complex cubic Ginzburg-Landau equation.


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