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On the Bifurcation Properties of Perturbed First-Order Differentiable Maps

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Abstract–A modulating input to a differentiable first-order system results in a limit set (or a 'belt') of points for the period-1 region in the steady state. The bifurcation properties of these 'belts' are numerically distinct from the fixed points of the equivalent unperturbed maps. It is shown that there exist bifurcation points of these belts and that their bifurcation diagram retains a structure similar to that of the fixed-point of the unperturbed case. It is proved that there exists an ordering between the bifurcation points of the 'belt' of the perturbed system and the fixed point of the unperturbed system. It is also shown that fixed intervals of a class of first-order perturbed maps correspond to fixed points of the second order map where the second order map is the class of first-order maps composed by itself.

1. Introduction

The bifurcation properties of unperturbed nonlinear difference equations are well documented [1]. Figure 1 shows the bifurcation diagram of the map given by

$$\phi_e(n) = \phi_e(n-1) + 2\pi. \ + A\sin[.\ n + .\ ] - 2\pi K_1(\sin(\phi_e(n-1)) + \delta) \mod(2\pi), \qquad (1)$$

for varying values of K_1. The map (1) formed the motivation for this work and describes a first-order Digital Phase-Locked Loop (DPLL) [3] where,
$A =$ Modulating amplitude, $(A = 0)$.
The map is of the form

$$F_q(x) = F(x) + q\,,\ q\ .\ [-A, A], \qquad (2)$$

and let

$$\mathrm{I} = \{F_q | q\ .\ [-A, A]\} \qquad (3)$$

where $F(x)$ is differentiable on some interval, I . For the case $A = 0$, the fixed point (white line) bifurcates in a pitchfork bifurcation at the point $K_1 = 0.333$ for the values given in figure 1. If the system has a modulated input $(A\ .\ 0)$ then the perturbed system is characterized in the steady state by a 'belt' of points as shown by the black region in figure 1. It is clear that the fixed point of the unperturbed system forms a 'skeleton structure' for this 'belt'. The size of the 'belt' depends on the size of A . Note that the term 'perturbed' as opposed to 'forced' is used to describe the class of systems given by (2), as the value q need not be deterministic. For example, q may be white noise restricted to the region $q\ .\ [-A, A]$. Figure 1 indicates that both the perturbed and unperturbed system (1) bifurcate in a similar way and that the point K_1 at which the 'belt' bifurcates, designated K_1^*, may not equal the bifurcation point of the fixed point of the unperturbed system. It is shown below that K_1^* in the unperturbed system never occurs before K_1^* in the perturbed system, provided A is small,

i.e. $$K_1^*(\delta, .\ , 0) = K_1^*(\delta, .\ , A) \ \text{for} \ \varepsilon > A > 0\,. \qquad (4)$$

More generally, it will be proved that there is an ordering at which the bifurcations of the perturbed and unperturbed systems occur under certain conditions.

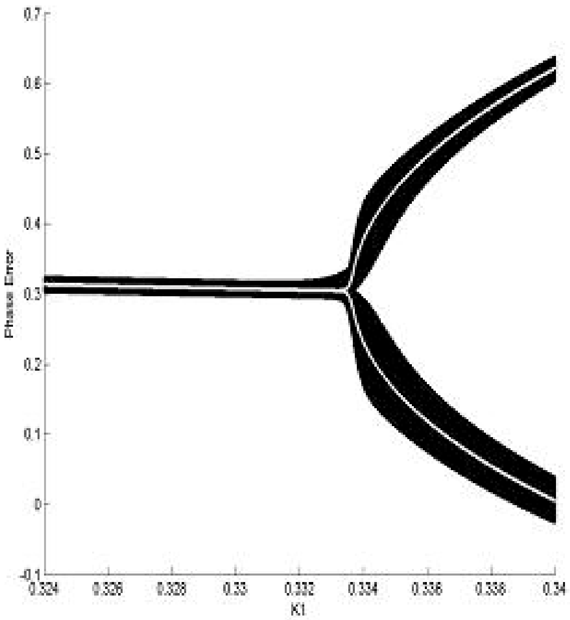

Figure 1: Bifurcation diagram of the system (1) with $. = 0.1$, $. = 0.005$, $\delta = 0$, $A = 0$ (white) and $A = 0.02$ (black).

2. Asymptotic Stability of the Fixed Interval

Definition 1— A Fixed Interval I^* *is a compact interval, which is invariant under* F_q *for all* $q\ .\ [-A, A]$.

Let F be continuous on I and A.1

$F_q : I\ .\ \ I$ for all $q\ .\ [-A, A]$. A.2

Then, given any closed interval $[l, u]\ .\ \ I$

$$\mathrm{I}\ ([l,u]) = \{y | y = F(x) + q, x\ .\ [l,u], q\ .\ [-A, A]\}, \qquad \text{A.3}$$

is a closed interval $[l', u']\ .\ \ I$.

Hence the class of maps I maps closed intervals in I to closed intervals in I.

Let ,

$$S_I = \left\{ \begin{bmatrix} l \\ u \end{bmatrix} .\ \ I^2 \middle| l = u \right\}. \qquad (5)$$

Given $\begin{bmatrix} l \\ u \end{bmatrix} .\ S_I$, let $\begin{bmatrix} l' \\ u' \end{bmatrix} .\ S_I$ be given by $[l', u'] = \mathrm{I}\ ([l,u])$, i.e. Associated with I there is a map $\tilde{F}$ from S_I to S_I such that $\begin{bmatrix} l' \\ u' \end{bmatrix} = \tilde{F}\begin{bmatrix} l \\ u \end{bmatrix}$.

Note: If $I^* = [l^*, u^*]\ \ .\ \ I$ is a fixed interval of I then $\mathrm{I}\ (I^*) = I^*$, i.e.

$\tilde{F}\left(\begin{bmatrix} l^* \\ u^* \end{bmatrix}\right) = \begin{bmatrix} l^* \\ u^* \end{bmatrix}$. Hence, $\begin{bmatrix} l^* \\ u^* \end{bmatrix}$ is a fixed point of $\tilde{F}$.

Moreover, if $\begin{bmatrix} l^* \\ u^* \end{bmatrix} .\ S_I$ is a fixed point of $\tilde{F}$ then $I^* = [l^*, u^*]$ is a fixed interval of I.

Definition 2— When F *satisfies (A.1)/(A.2) then a fixed interval* $I^* = [l^*, u^*]\ \ .\ \ I$ *of* I *is asymptotically stable if the corresponding fixed point of* $\tilde{F}$ *is asymptotically stable.*

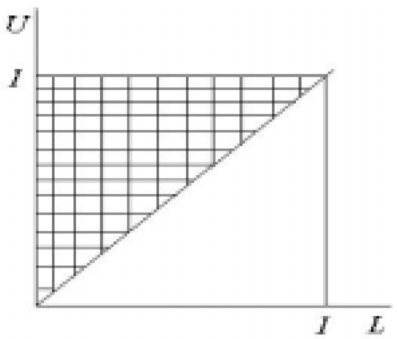

Figure 2: Invariant region S_I given by the meshed section.

Main Theorem

Given F , F_q *and* I *as in (2-3), if*

(i) F *is differentiable on a compact interval* I.

(ii) $F_q : I\ .\ \ I$ *for all* $q\ .\ [-A, A]$,

(iii) $|F'(x)| = .\ < 1$ *for all* $x\ .\ \ I$.

(iv) F *is strictly monotone on* I.

Then

1. For each $q\ .\ [-A, A]$ *there exists a unique fixed point* $x_q^*\ .\ \ I$ *of* F_q.

2. There exists a unique asymptotically stable fixed interval $I^* = [l^*, u^*]\ \ .\ \ I$ *of* I.

3. If F *is strictly monotone increasing then* $I^* = \{x_q^* | q\ .\ \ [-A, A]\}$.

Proof (1)

Lemma 1

Under the conditions of the main theorem there exists a unique fixed point $x_q^*\ .\ \ I$ *of* F_q *for all* $q\ .\ [-A, A]$.

1 follows directly from lemma 1.

Proof (2)

Case 1: F *is strictly monotone increasing on* I.

Then by A.3 and by the definition of $\tilde{F}$,

$$\tilde{F}\begin{bmatrix} l \\ u \end{bmatrix} = \begin{bmatrix} l' \\ u' \end{bmatrix} = \begin{bmatrix} F(l) - A \\ F(u) + A \end{bmatrix}. \qquad (6)$$

Lemma 2

Under the conditions of the main theorem, $\tilde{F}$ *of (6) has a unique fixed point in* S_I.

Let $\begin{bmatrix} l^* \\ u^* \end{bmatrix} .\ S_I$ equal the unique fixed point in S_I of $\tilde{F}$.

Then $[l^*, u^*]\ \ .\ \ I$ is a unique fixed interval of I.

The Jacobian of $\tilde{F}$ at $\begin{bmatrix} l^* \\ u^* \end{bmatrix} = J = \begin{bmatrix} F'(l^*) & 0 \\ 0 & F'(u^*) \end{bmatrix}$ (7)

Moreover by (iii),

$$|F'(l^*)| < 1 \text{ and } |F'(u^*)| < 1. \quad (8)$$

since $l^*, u^* .\ I$. Hence 2 follows by definition 2.

Case 2: F is strictly monotone decreasing on I.

$$\tilde{F}\begin{bmatrix} l \\ u \end{bmatrix} = \begin{bmatrix} l' \\ u' \end{bmatrix} = \begin{bmatrix} F(u) - A \\ F(l) + A \end{bmatrix}. \quad (9)$$

By Lemma 2, the map $\tilde{F}$ has a unique fixed point in S_I.

Let $\begin{bmatrix} l^* \\ u^* \end{bmatrix} .\ S_I$ equal the unique fixed point in S_I of $\tilde{F}$.

Then $[l^*, u^*]\ .\ I$ is a unique fixed interval of I .

Using a similar procedure as in case 1 the Jacobian matrix, J, is found to be equal to

$$J = \begin{bmatrix} 0 & F'(u^*) \\ F'(l^*) & 0 \end{bmatrix}, \quad (10)$$

Moreover by (iii),

$$|F'(l^*)| < 1 \text{ and } |F'(u^*)| < 1. \quad (11)$$

since $l^*, u^* .\ I$. Hence 2 follows by definition 2.

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Proof (3)

As x_q^* is a unique fixed point in I of F_q then

$$x_q^* = F(x_q^*) + q. \quad (12)$$

The derivative of (12) with respect to q is equal to

$$x_q^{*\prime} = F'(x_q^*) x_q^{*\prime} + 1. \quad (13)$$

i.e.

$$x_q^{*\prime} = \frac{1}{1 - F'(x_q^{*\prime})} = \frac{1}{1 - .} > 0, \qquad \text{by (iii) as } . < 1 \quad (14)$$

which implies that x_q^* is strictly monotone increasing and continuous for all $q .\ [-A, A]$. Therefore,

$I^+ = \{x_q^* | q .\ [-A, A]\} = [x_{-A}^*, x_A^*]$ is a closed interval in I.

$$\mathrm{I}\ (I^+) = \{y | y = F(x) + q, x .\ I', q .\ [-A, A]\} \quad (15)$$

$$= \{y | y = F(x_{q_1}^*) + q,\ q_1, q .\ [-A, A]\} \quad (16)$$

$$= \{y | y = x_{q_1}^* - q_1 + q,\ q_1, q .\ [-A, A]\} \quad (17)$$

case 1: $F'(x_q^*) > 0$

$$x_q^{*\prime} > 1. \quad (18)$$

This implies that

$x_{q_1}^* - q_1$ must be strictly monotone increasing in q_1.

$$x_{q_1}^* - q_1 + A = x_A^* - A + A \quad (19)$$

$$x_{q_1}^* - q_1 - A = x_{-A}^* - A + A \quad (20)$$

therefore

$$\mathrm{I}\ (I^+) = (I^+) \quad (21)$$

.

3 Corollary 1

Under the conditions of the main theorem if F is continuously twice differentiable on I and $F''(x)$ is sign definite on I^*, then the region of stability of the 'belt' of the perturbed system contains the region of stability of the fixed point of the unperturbed system.

Consider the example of the first-order DPLL, (1). Using the values as given in figure 1, the fixed point of (1) is equal to

$$x_0^* = \sin^{-1}\left(\frac{0.1}{K_1}\right) \quad (22)$$

Considering the value $K_1 = 0.32$, the fixed point is equal to $x_0^* = 0.3178$. Let,

$$F(x) = x + 0.2\pi - 2\pi K_1 \sin(x) \text{ for all } x .\ [-3,3]$$

Condition (i) holds as F is differentiable on $[-3,3]$.

Let ,

$$I = [l, u] .\ [-3,3] \quad (23)$$

Condition (iii) implies

$$|F'(x)| = |1 - 2\pi K_1 \cos(x)| < 1 \text{ therefore,}$$
$$2\pi(0.32)\cos(l) < 2$$

$$\text{therefore, } l = 0.1028. \quad (24)$$

Condition (iv) says that $F'(x) < 0$ for all $x\ .\ [l,u]$ Therefore,

$$2\pi(0.32)\cos(u) > 1$$
$$\text{therefore, } u < 1.05. \quad (25)$$

Condition (ii) implies

$$F_q(x) = x + 0.2\pi - 2\pi(0.325)\sin(x) + q \quad (26)$$
$$= [x - 2\pi(0.325)\sin(x)] + 0.2\pi + A \quad (27)$$
$$= [l - 2\pi(0.325)\sin(l)] + 0.2\pi + 0.01 = u \quad (28)$$
$$= [u - 2\pi(0.325)\sin(u)] + 0.2\pi - 0.01 = l \quad (29)$$

Let $I = [0.166, 0.48]$. $F_q(I) = [0.1699, 0.4721]$, therefore, the interval $I = [0.166, 0.48]$. I^* satisfies condition (ii) and F is sign definite on $I\ .\ I^*$.

$F''(x) > 0$ for all $x\ .\ [0.166, 0.48]$. Therefore, $F''(x)$ is sign definite on the fixed interval I^* and there is an ordering for which the perturbed and the unperturbed systems (1) bifurcate,

i.e. x_0^* is asymptotically stable if I^* is asymptotically stable.

4. Conclusion

This paper has shown that bifurcation points of the class of maps described by (2) exist that retain a similar structure to the bifurcation diagram of the equivalent unperturbed map. This paper also presented the proof that the region of stability of the ‘belt’ of the perturbed system contains the region of stability of the fixed point of the unperturbed system. If the function describing the system, (2), is continuously twice differentiable and sign definite on the fixed interval and the conditions of the main theorem hold then there is an ordering for which the perturbed system and the unperturbed system bifurcate. Finally, it was shown that there exists a relationship between the map $\tilde{F}$ and the class of maps I , namely: asymptotically stable fixed points of the map $\tilde{F}$ correspond to asymptotically stable fixed intervals of I . Therefore, the analysis can be treated as a fixed-point analysis.

5. Appendix

Proof of Lemma 1

$$x_1, x_2\ .\ \ I, \quad (30)$$
$$|F_q(x_1) - F_q(x_2)| = |F(x_1) - F(x_2)| \quad (31)$$
$$= |F'(c)(x_1 - x_2)| = .\ |x_1 - x_2|, \text{ by (iii)} \quad (32)$$

where $c\ .\ [x_1, x_2]\ .\ \ I$.

$$|F'(x)| = .\ < 1 \text{ for all } x\ .\ I. \quad (33)$$

Moreover, $F_q : I\ .\ \ I$ by (ii). Since I is compact the result follows by the contraction mapping principle.

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Proof of Lemma 2

$$\tilde{F}\begin{bmatrix} l \\ u \end{bmatrix} = \begin{cases} \begin{bmatrix} F(l) - A \\ F(u) + A \end{bmatrix} \text{for case 1.} \\ \begin{bmatrix} F(u) - A \\ F(l) + A \end{bmatrix} \text{for case 2.} \end{cases} \text{, where } \tilde{F} : S_I\ .\ \ S_I.$$

$$\left\| \tilde{F}\begin{bmatrix} l_1 \\ u_1 \end{bmatrix} - \tilde{F}\begin{bmatrix} l_2 \\ u_2 \end{bmatrix} \right\|_8 = \begin{cases} \left\| \begin{matrix} F(l_1) - F(l_2) \\ F(u_1) - F(u_2) \end{matrix} \right\|_8 \text{ for case 1} \\ \left\| \begin{matrix} F(u_1) - F(u_2) \\ F(l_1) - F(l_2) \end{matrix} \right\|_8 \text{ for case 2} \end{cases} \quad (34)$$

$$= Max\{|F(u_1) - F(u_2)|, |F(l_1) - F(l_2)|\} \quad (35)$$
$$= Max\{.\ |u_1 - u_2|, .\ |l_1 - l_2|\} \quad (36)$$

$$= .\ \left\| \begin{bmatrix} l_1 - l_2 \\ u_1 - u_2 \end{bmatrix} \right\|_8 = .\ \left\| \begin{bmatrix} l_1 \\ u_1 \end{bmatrix} - \begin{bmatrix} l_2 \\ u_2 \end{bmatrix} \right\|_8 . \quad (37)$$

By the contraction mapping principle $\tilde{F}$ takes points in S_I to points in S_I.

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