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Coupling-enhanced deterministic stochastic resonance

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Abstract—We show that deterministic stochastic resonance (DSR) can be enhanced by coupling of chaotic oscillators and there is an optimal coupling strength which maximizes the resonance response. We study periodic-forced chaotic oscillators coupled to each other. The phases of the oscillators and periodic forcing synchronize with each other when the force strength is large. When we set the force strength below a critical value, the phase synchronization occasionally fails, and we can observe intermittent slips in the phase differences. The strength of the coupling between the oscillators is another bifurcation parameter that has a critical point between asynchronous and synchronous phase slip state. When the coupling is larger than the critical value, the DSR effect is enhanced by synchronized slips. Since the increasing of coupling strength also controls the average rate of slips, that is the internal fluctuation, we can see an optimal coupling strength which maximizes the DSR response. To our knowledge, this is the first report of enhanced DSR by coupling in a deterministic system.

1. Introduction

Noise-induced effects in nonlinear systems have recently received considerable attention. In particular, stochastic resonance (SR) [1, 2] has been studied in various systems. The resonance response of a noisy nonlinear system to a subthreshold signal can be optimized by noise intensity. Several studies have reported SR-like behavior also in chaotic systems, both numerically and experimentally [3, 4, 5, 6, 7]. These resonance behaviors are called deterministic SR (DSR), and it is often interpreted that chaos intrinsically generates the noise in the SR scenario. In some DSR studies, the researchers insisted that crises have an important relationship with SR. For example, DSR has been reported in a deterministic chaotic map that shows a *two attractors merging crisis* at a critical point [3, 5, 6] and DSR is observed with synchronization of the attractor switching and an injected periodic signal. SR can be found in bistable systems not only of fixed points but of any kind of attractors, even chaos.

We have reported DSR in phase slips that occur when phase synchronization fails in a sinusoidally forced Rössler oscillator [8, 9]. When the forcing frequency is close to the natural frequency of a Rössler oscillator, the two phases are synchronized in the sense that the difference between these phases stays within a given region [10, 11, 12]. However, when the difference between their frequencies increases the phase synchronization breaks down and intermittent slips in the phase difference occur. These intermittent phase slips, where the phase difference quickly changes by 2π, are considered to be jumps to another chaotic attractor in the phase space. The transition between phase synchronization and phase slips is caused by an *unstable-unstable pair bifurcation crisis* [13]. The DSR mechanism can be explained in terms of the synchronization between the chaotic attractor switching and the modulation of a bifurcation parameter by an injected periodic signal.

Here we demonstrate enhanced DSR resulting from coupling between forced Rössler oscillators. Similar to previous studies on coupled resonators [14, 15, 16, 17], coupling synchronizes resonator behavior and the resonance response is enhanced as a result. This paper is organized as follows. In the next section, we introduce the coupled forced Rössler oscillators. In Sect. 3 we show the enhancement of DSR when coupling is introduced. Section 4 contains concluding remarks.

2. Model system

The DSR using phase slips has been studied in Ref. [8, 9] in detail. Here we focus on the following coupled resonators to study the enhanced DSR achieved by coupling.

$$\begin{aligned} \dot{x_i} &= s_i(-\nu_i y_i - z_i), \\ \dot{y_i} &= s_i(\nu_i x_i + a y_i) + K[1+\epsilon\sin(\omega t)]\sin(\Omega t) \\ &\quad + \sum_{j=1}^{N} \frac{C}{N}(y_j - y_i), \qquad (1) \\ \dot{z_i} &= s_i(b + z_i(x_i - c)). \end{aligned}$$

Here $s_i = 1 + \alpha(r_i^2 - \overline{r}^2)$, where $r_i = \sqrt{x_i^2 + y_i^2}$, and $\overline{r}$ and α are constants whose meaning is explained later.

We set the system parameters $a = 0.2$, $b = 0.2$, and $c = 4.8$. The periodic forcing term $K\sin(\Omega t)$ is fixed as $K = 0.07$ and $\Omega = 1.077$ throughout this paper. The signal term $\epsilon \sin(\omega t)$ is also fixed as $\epsilon = 0.05$ and $\omega = 6.0 \times 10^{-4}$. We define the frequency average ν_m and the difference $\Delta\nu$ among the natural frequencies ν_i, $i = 1, \ldots, N$ of each oscillator as $\nu_m = \frac{1}{N}\sum_{i=1}^{N} \nu_i$ and $\Delta\nu = \frac{1}{N}\sum_{i=1}^{N} |\nu_i - \nu_m|$. In this paper, the frequency average ν_m and the coupling strength C are used as control parameters.

Here we focus on the phases of the chaotic oscillators. The projection of the Rössler attractor on the x-y plane forms a ring around the origin. To describe the phase θ_i of the chaotic oscillator, we define $\theta_i = \arctan \frac{y_i}{x_i}$. We then define the phase difference between the Rössler oscillator and the forcing as $\phi_i = \theta_i - \Omega t$.

Although it is not so significant in this paper, we mention the meaning of factor s_i [8, 9]. The factor s_i enlarges the variation of the angular velocity, driven by the variation of the amplitude. Therefore, as α increases, the variation of the angular velocity increases. Phase slips occur more frequently as α increases. In the following, we set $\alpha = 0.002$. $\overline{r}$ is the average r value for an ordinary Rössler oscillator ($N = 1$, $\alpha = K = C = 0$, $\nu = 1$).

For the parameter regime studied in this paper, the oscillator phase θ_i intermittently breaks synchronization with the phase Ωt of the periodic forcing and the phase difference occasionally jumps 2π. We observe DSR in the behavior of this phase slip. We consider each forced Rössler oscillator as a unit resonator. In this section, we choose $N = 2$ for explanation as the simplest coupled resonator system. As in the single resonator case described in Ref. [8, 9], the difference between the forcing frequency Ω and the average frequency ν_m of the Rössler oscillator governs the average rate of phase slips. We consider the average rate of phase slips to be the strength of the internal fluctuations, or the stochastic "noise" in the DSR scenario. In this paper, we fix Ω, and use ν_m as the parameter characterizing the strength of this "noise".

To study phase slip synchronization, we focus on the resonator phase difference $\Delta\phi$ between ϕ_1 and ϕ_2, that is, $\Delta\phi = \phi_1 - \phi_2 = \theta_1 - \theta_2$. For now, we consider the case without the signal ($\epsilon = 0$) unless otherwise stated. If the coupling strength $C = 0$, the phase slips of each forced oscillator occur intermittently and independently of each other. In this situation, ϕ_i occasionally jumps 2π independently, while $\Delta\phi$ increases and decreases by 2π independently. Intermittent synchronization of phase slips also occurs for non-zero coupling strength C when the coupling strength C is too small, as in Fig. 1(a) and (c). When the coupling strength C exceeds a critical value, the phase slips of each forced oscillator occur intermittently but the slips of both resonators are synchronized with each other. Although phase slips occur with respect to the forcing, the resonator phase difference $\Delta\phi$ is always within a certain range, as in Fig. 1(b) and (d) [18].

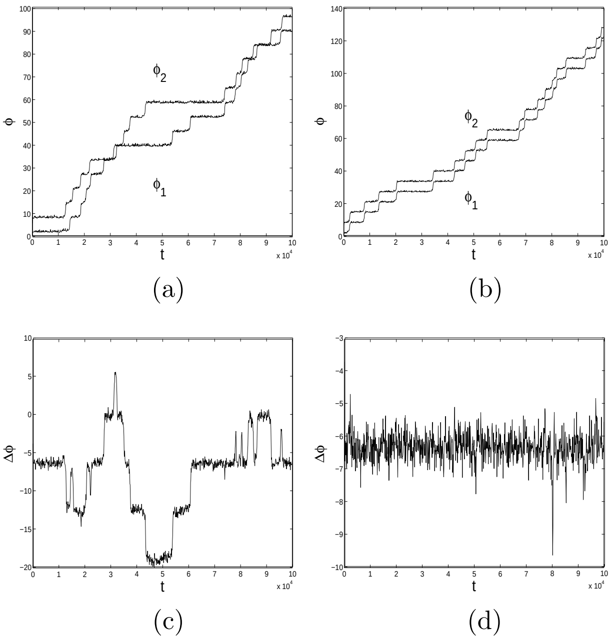
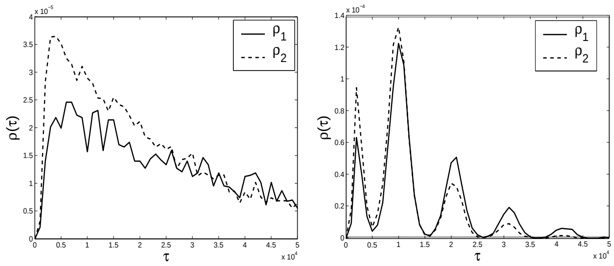


Figure 1: Phase slips in two coupled resonators without signal modulation. The mean frequency parameter is $\nu_m = 1.004$ and the difference is $\Delta\nu = 0.00005$. Figures (a) and (c) shows the phase difference and the resonator phase difference respectively when $C = 0.002$. Figures (b) and (d) also shows the phase difference and the resonator phase difference respectively when $C = 0.006$.

3. Coupling enhancement of DSR

This section provides numerical results on DSR. We consider $N = 2$, unless otherwise stated. The interslip interval distributions (ISIDs) ρ_i ($i = 1, 2$) for each resonator are shown in Fig.2. The frequency difference $\Delta\nu$ results in different forms of ISIDs. When the forcing is modulated by an external periodic signal with frequency ω, the distribution appears modulated with peaks at integer multiples τ_n ($= 2\pi n/\omega$, $n = 1, 2, \ldots$) of the modulating signal period. SR is observed as change in ρ_i [19]. To quantify the strength of the resonance response, we use the difference in the ISID probability at τ_1, with and without a signal, i.e. $\Delta\rho_i = \rho_{i,\epsilon>0}(\tau_1) - \rho_{i,\epsilon=0}(\tau_1)$.

In the case of coupled resonators, the behavior of the ISIDs, the exponential decay without a signal and the exponential decay of the multipeak envelope with a signal, are similar to the single resonator result. The difference between the distributions become closer

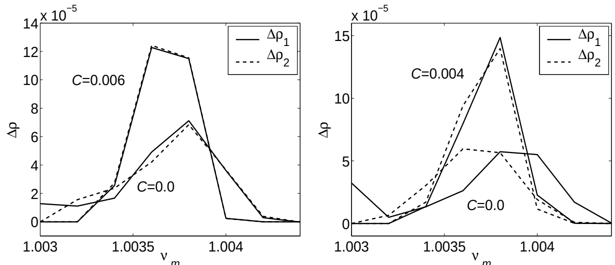


(a) (b)

Figure 2: Probability distribution ρ_i of the interslip time intervals τ. The frequency parameter is $\nu = 1.0038$ and the difference is $\Delta\nu = 0.00005$. The 1st resonator is shown by lines. The 2nd resonator shown by dotted lines. Figure (a) shows the distribution without a signal ($\epsilon = 0$) and Fig. (b) shows the distribution with signal ($\epsilon = 0.05 > 0$).

when the coupling strength increases, both with and without a signal ($\epsilon = 0$ and > 0). When the slips are perfectly synchronized with each other with sufficient coupling strength, the ISIDs for each resonator overlap and are coincident.

Figure 3 (a) shows the resonance strength $\Delta\rho$ versus ν_m when $\Delta\nu = 0$. $\Delta\rho$ has a maximum value at a

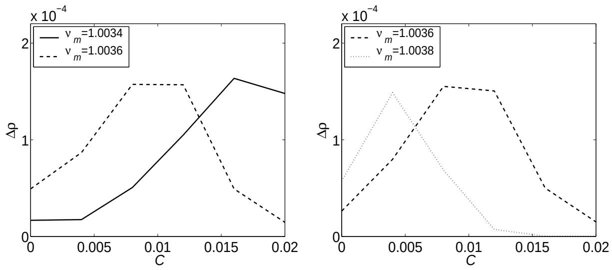


(a) (b)

Figure 3: Resonance strength $\Delta\rho$ as a function of parameter ν_m. Here ν_m represents the strength of the internal fluctuation. The 1st resonator is shown by lines. The 2nd resonator shown by dotted lines. Figure (a) shows $\Delta\rho$ when $\Delta\nu = 0$ and Fig. (b) shows $\Delta\rho$ when $\Delta\nu = 0.0001(> 0)$.

particular value of ν_m, which represents the strength of the internal fluctuation. This is similar to the single resonator result [8, 9]. The maximum $\Delta\rho$ value with $C > 0$ is larger than that with $C = 0$. From this figure it can be seen that the coupling strength C can enhance the resonance response. If the frequency difference of the oscillator is zero ($\Delta\nu = 0$), even small coupling can synchronize the slips that occur between the phases of each forced oscillator and the periodic forcing. This is related to the enhancement of the resonance response. Similar to this result, Fig. 3 (b) shows that the coupling C can also enhance the resonance response when $\Delta\nu > 0$. When the difference $\Delta\nu$ is larger than zero, the size of the coupling C that is required to synchronize the phase slips is larger than when $\Delta\nu = 0$. For both cases, when the coupling is larger than a certain value, the resonance strength $\Delta\rho_1$ and $\Delta\rho_2$ curves of the two oscillators become closer as seen in Fig. 3 (a) and (b), and finally overlap. This is because the phase slips of both resonators are synchronized by the coupling between the oscillators.

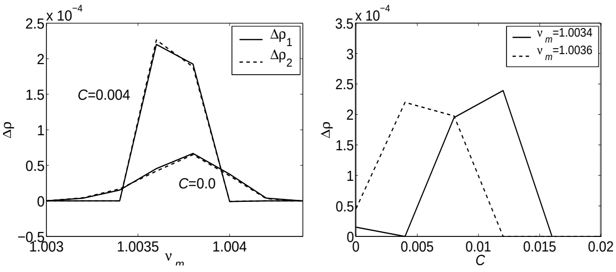
Figure 4 shows the resonance strength $\Delta\rho_1$ as a function of the coupling strength C at a fixed ν_m. It



(a) (b)

Figure 4: Resonance strength$\Delta\rho_1$ as a function of coupling strength C at a fixed ν_m. Figure (a) shows $\Delta\rho$ when $\Delta\nu = 0$ and Fig. (b) shows $\Delta\rho$ when $\Delta\nu = 0.0001(> 0)$.

can be seen that there is an optimal coupling strength C_{opt} which produces the maximal resonance response in the $\Delta\rho_1$ value. This optimality is qualitatively discussed later in Sect. 4. Figure 5 shows that similar coupling-enhanced response occurs for a larger number of oscillators, $N = 16$, when $\Delta\nu = 0$. There is an



(a) (b)

Figure 5: Resonance strength $\Delta\rho$. Figure (a) shows $\Delta\rho$ as a function of parameter ν_m and Fig. (b) shows $\Delta\rho$ as a function of coupling strength C at a fixed ν_m.

optimal ν_m in Fig.5 (a) and there is an optimal coupling strength C_{opt} in Fig.5 (b), which produces the maximal resonance response.

4. Conclusion

We have shown numerically that coupling can enhance the deterministic stochastic resonance (DSR) response. In this sense, our result on enhanced resonance is consistent with the results of other studies of coupling-enhanced SR [14, 15, 16, 17]. However, to our knowledge, this is the first report of enhanced DSR by coupling in a deterministic system.

This enhancement can be described qualitatively as follows. When an external signal modulates the forcing in this system, the DSR effect is seen as statistical synchronization between the intermittent phase slips for each resonator and the external signal, manifest as resonance peaks in the interslip interval distribution (ISID). When the coupling is larger than a critical value, phase slips between resonators are synchronized with each other, causing the ISIDs of the intermittent phase slips in the resonators to be similar even when the oscillator frequencies are different, and the DSR effect is enhanced.

Increasing of coupling strength increases the average rate of slips, that is the internal fluctuation. In the case of standard SR, the increasing of coupling strength only suppresses effective noise intensity. In this sense, there is a difference in the effect of the coupling strength on fluctuations in the standard SR and DSR in the coupled forced Rössler oscillators. Detail behavior about effects of coupling will be reported elsewhere.

References

[1] A. R. Bulsara and L. Gammaitoni, "Tuning in to noise," *Phys. Today*, vol.49, pp.39–45, 1996.

[2] L. Gammaitoni, P. Hänggi, P. Jung and F. Marchesoni, "Stochastic resonance," *Rev. Mod. Phys.*, vol.70, pp.223–287, 1998.

[3] V. S. Anishchenko, A. B. Neiman and M. A. Safanova, "Stochastic resonance in chaotic systems," *J. Stat. Phys.*, vol.70, pp.183–196, 1993.

[4] G. Nicolis, C. Nicolis and D. McKernan, "Stochastic resonance in chaotic dynamics," *J. Stat. Phys.*, vol.70, pp.125–139, 1993.

[5] S. Sinha and B. K. Chakrabarti, "Deterministic stochastic resonance in a piecewise linear chaotic map," *Phys. Rev. E*, vol.58, pp.1298–1304, 1998.

[6] S. Mizutani, H. Suetani, K. Arai and K. Yoshimura, "Deterministic stochastic resonance in chaotic diffusion," *Proc. SPIE's First International Symposium on Fluctuations and Noise*, vol.5114, pp.488–497, 2003.

[7] H. Yamazaki, T. Yamada and S. Kai, "Can stochastic resonance lead to order in chaos?," *Phys. Rev. Lett.*, vol.81, pp.4112–4115, 1998.

[8] K. Arai, K. Yoshimura and S. Mizutani, "Dynamical origin of deterministic stochastic resonance," *Phys. Rev. E*, vol.65, pp.015202: 1–4, 2002.

[9] K. Arai, S. Mizutani and K. Yoshimura, "Deterministic stochastic resonance in a Rössler oscillator," *Phys. Rev. E*, vol.69, pp.026203: 1–10, 2004.

[10] M. G. Rosenblum, A. S. Pikovsky and J. Kurths, "Phase synchronization of chaotic oscillators," *Phys. Rev. Lett.*, vol.76, pp.1804–1807, 1996.

[11] A. Pikovsky, G. Osipov, M. Rosenblum, M. Zaks and J. Kurths, "Attractor-repeller collision and eyelet intermittency at the transition to phase synchronization," *Phys. Rev. Lett.*, vol.79, pp.47–50, 1997.

[12] E. Rosa Jr., E. Ott and M. H. Hess, "Transition to phase synchronization of chaos," *Phys. Rev. Lett.*, vol.80, pp.1642–1645, 1998.

[13] E. Ott, "Chaos in dynamical systems," *Cambridge University Press*, 1993.

[14] A. Neiman and L. Schimansky-Geier, "Stochastic resonance in two coupled bistable systems," *Phys. Lett. A*, vol.197, pp.379–386, 1995.

[15] P. Jung and G. Mayer-Kress, "Spatiotemporal stochastic resonance in excitable media," *Phys. Rev. Lett.*, vol.74, pp.2130–2133, 1995.

[16] J. F. Lindner, B. K. Meadows, W. L. Ditto, M. E. Inchiosa and A. R. Bulsara, "Array enhanced stochastic resonance and spatiotemporal synchronization," *Phys. Rev. Lett.*, vol.75, pp.3–6, 1995.

[17] M. E. Inchiosa and A. R. Bulsara, "Coupling enhanced stochastic resonance in nonlinear dynamic elements driven by a sinusoid plus noise," *Phys. Lett. A*, vol.200, pp.283–288, 1995.

[18] K. Arai and S. Mizutani, "Synchronization of phase slips in chaotic map," *Proc. SPIE's Second International Symposium on Fluctuations and Noise*, vol. 5471, pp. 492–503, 2004.

[19] M. H. Choi, R. F. Fox and P. Jung, "Quantifying stochastic resonance in bistable systems: response vs residence-time distribution functions," *Phys. Rev. E*, vol.57, pp.6335–6343, 1998.