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SAMPLED-DATA CONTROL OF NONLINEAR SYSTEMS USING STATE-SPACE RECURSIVE LEAST SQUARES

Mohammad Bilal Malik, Khalid Munawar
College of Electrical and Mechanical Engineering
National University of Sciences and Technology, Pakistan.

ABSTRACT

State-space recursive least-squares (SSRLS) is a new addition to the family of RLS adaptive filters. Superior tracking performance and state-space formulation are the strengths of SSRLS. In this paper, we show that certain forms of SSRLS act as approximate discrete differentiators. High-gain SSRLS has a disturbance rejection property. This analogy between continuous high-gain observers and high-gain SSRLS provides a rationale to use SSRLS as discrete high-gain observer. This development enables us to design a sampled data control for a class of continuous nonlinear systems. An example of controlling an inverted pendulum illustrates the ideas presented in this framework.

1 INTRODUCTION

The theory and concept of state-space recursive least-squares (SSRLS) was introduced in ([11]). As a natural extension of SSRLS, we have developed SSRLS with adaptive memory ([11], [12]). SSRLS gives the designer a freedom to choose an appropriate signal model. Therefore, SSRLS exhibits superior tracking characteristics as compared to the standard RLS ([11], [12]). Its state-space formulation makes it especially suited to be used as a state estimator in control systems.

High-gain observers have played an important role in the design of output feedback control of nonlinear systems. Khalil and coworkers pioneered this concept and have presented a number of results ([1], [2], [5], [8]). Other major contributors to this theory include Isidori, Kokotovic, Saberi and their coworkers respectively ([7], [10], [14], [15]) etc. and the references therein). Lately Khalil and Dabroom ([3], [4]) extended this concept to sampled-data control systems. Their approach is to design the observer and controller in continuous time and then discretize the observer and/or controller. In this paper, we explore the possibility of using SSRLS as a discrete high-gain observer. The observer and controller are designed solely in discrete domain, which simplifies the design procedure.

We begin by showing that certain forms of SSRLS act as approximate discrete differentiators. Subsequently we prove the disturbance rejection property of high-gain SSRLS. This analogy between high-gain observers and high-gain SSRLS provides a rationale to use SSRLS as discrete high-gain observer. As the plant is taken to be continuous nonlinear system, we have to discretize it in order to design the controller and observer. For this purpose we use a modified triangular hold equivalent that suits our development. The controller design is done for a class of nonlinear systems that are feedback linearizable. An example of controlling an inverted pendulum illustrates the ideas presented in this paper.

2 STATE-SPACE MODEL

Consider the following unforced discrete time system.

$$\begin{aligned} x[k+1] &= Ax[k] \\ y[k] &= Cx[k] + v[k] \end{aligned} \tag{2.1}$$

where $x \in R^n$ are the process states, $y \in R^m$ is the output vector and $v[k]$ is the observation noise. We assume that the pair (A, C) is l-step observable [13] and A is invertible.

3 PREVIEW OF SSRLS

Suppose that observations $y[k]$ start appearing at time $k=1$. According to SSRLS ([11]) state estimate $\hat{x}[k]$ is given as

$$\hat{x}[k] = \bar{x}[k] + K[k]\varepsilon[k], \quad k \geq 1 \tag{3.1}$$

where $\bar{x}[k] = A\hat{x}[k-1]$ is the predicted state estimate. The prediction error, which is also referred to as innovations, is defined as

$$\varepsilon[k] = y[k] - \bar{y}[k] \tag{3.2}$$

with $\bar{y}[k] = C\bar{x}[k]$ as the predicted output. Observer gain $K[k]$ is determined by the method of least squares and is given by

$$K[k] = P[k]C^T \tag{3.3}$$

where $P[k]$ is the solution of the Riccati equation for SSRLS. We define $0 < \lambda \leq 1$ as the forgetting factor [11].

3.1 Riccati Equation of SSRLS

$P[k]$ is recursively updated by the following equation.

$$\begin{aligned} P[k] &= \lambda^{-1}AP[k-1]A^T - \lambda^{-2}AP[k-1]A^TC^T \times \\ &\left[I + \lambda^{-1}CAP[k-1]A^TC^T\right]^{-1} CAP[k-1]A^T \end{aligned} \tag{3.4}$$

Initial condition is $P[0]$, which is preferably positive definite as would become clear later in this paper.

3.2 Recursive Update of $Q[k]$

Define $Q[k] = P^{-1}[k]$, which can be recursively updated by

$$Q[k] = \lambda A^{-T}Q[k-1]A^{-1} + C^TC \tag{3.5}$$

4 PREVIEW OF STEADY-STATE SSRLS

If the following limit exists

$$\lim_{k\to\infty} P[k] = P \tag{4.1}$$

then SSRLS settles down asymptotically to an LTI filter, which we term as steady-state SSRLS. The observer gain is not a function of time and is given by

$$K = PC^T \tag{4.2}$$

Corresponding to (3.4), in this case we have the algebraic Riccati equation

$$\begin{aligned} P &= \lambda^{-1}APA^T - \\ &\lambda^{-2}APA^TC^T\left[I + \lambda^{-1}CAPA^TC^T\right]^{-1} CAPA^T \end{aligned} \tag{4.3}$$

Under the following additional constraint

$$\sqrt{\lambda} < \min \left| Eigenvalues(A) \right| \tag{4.4}$$

(3.5) is transformed into discrete Lyapunov equation [11]

$$Q = \lambda A^{-T} Q A^{-1} + C^T C \tag{4.5}$$

Observability of (A, C) ensures $Q > 0$.

5 SSRLS AS AN APPROXIMATE DISCRETE DIFFERENTIATOR

In this section we show that certain forms of SSRLS are able to act as approximate discrete differentiator. We are able to draw an analogy between these forms of SSRLS and backward difference. This is in accordance with the fact that backward difference and SSRLS are both causal. Consider the constant velocity model [12]. Its transfer function has the property

$$\lim_{\lambda \to 0} H(z) = \begin{bmatrix} 1 \\ \dfrac{1 - z^{-1}}{T} \end{bmatrix} \tag{5.1}$$

The second entry is the well-known backward difference. Similarly transfer function [12] corresponding to the constant acceleration model has the property

$$\lim_{\lambda \to 0} H(z) = \begin{bmatrix} 1 \\ \dfrac{3 - 4z^{-1} + z^{-2}}{2T} \\ \dfrac{\left(1 - z^{-1}\right)^2}{T^2} \end{bmatrix} \tag{5.2}$$

The second row is the three-point difference formula [9] that has been used in numerical differentiation. The third row can be recognized as the second backward difference formula. Continuing in a similar fashion, it can be shown that SSRLS can be used to estimate differences of arbitrary orders. n^{th} order model [12] serves as the most general framework in this context. All of these models are neutrally stable.

6 DISTURBANCE REJECTION PROPERTY OF HIGH-GAIN SSRLS

When the forgetting factor λ is small the observer gain is large. Under this condition we term the filter as high-gain SSRLS. Such a filter has ability to attenuate the effect of unwanted disturbances. Consider a perturbed version of our original system (2.1) in absence of observation noise as follows

$$\begin{aligned} x[k+1] &= A\,x[k] + w[k] \\ y[k] &= C\,x[k] \end{aligned} \tag{6.1}$$

where $w \in R^n$ is a deterministic disturbance signal. The estimation error can be written as

$$\begin{aligned} e[k] &= x[k] - \hat{x}[k] \\ &= Fe[k-1] + w[k-1] \end{aligned} \tag{6.2}$$

where $\hat{x}[k]$ is the usual steady-state SSRLS estimate and we have defined $F = A - KCA$.

Theorem 6.1 [12]

The estimation error $e[k]$ asymptotically satisfies the following bound

$$\lim_{k \to \infty} \left\| e[k] \right\| \le \frac{\alpha}{1 - \rho(F)} \sup_{k \ge 0} \left\| w[k] \right\| \tag{6.3}$$

where α is a positive constant and $\rho(F)$ is the spectral radius of F.

6.1 Remarks

Inequality (6.3) can be transformed into

$$\lim_{k \to \infty} \left\| e[k] \right\| \le \frac{\alpha}{1 - \lambda} \sup_{k \ge 0} \left\| w[k] \right\| \tag{6.4}$$

for neutrally stable systems [12]. It is clear from (6.4), that smaller the λ, lesser will be the influence of the disturbance on estimation error. However, perfect disturbance attenuation is not achieved even when $\lambda \to 0$. This restriction is due to finite sampling time. Although not apparent from this discussion, decreasing the sampling time helps alleviate this limitation.

7 NONLINEAR CONTROL SYSTEMS

We briefly discuss a well-known control problem [8] that addresses a class of single-input single-output nonlinear systems. This section sets a ground for the later development. Our emphasis will be on the systems that can be represented in the normal form [8] as follows

$$\begin{aligned} \dot{z} &= f_o\left(z, x\right) \\ \dot{x} &= A_c x + B_c \frac{1}{\beta(z,x)} \left[u - \alpha(z,x) \right], \quad \beta(z,x) \ne 0 \\ y &= C_c x \end{aligned} \tag{7.1}$$

where n is the order of the system, $x \in R^{\rho}, z \in R^{n-\rho}$ are the states, y is the output, u is the input and $\rho \le n$ is the relative degree of the system. $\alpha(z,x)$ and $\beta(z,x)$ are scalar function of the states x and z and are assumed to be locally Lipschitz. The matrices A_c, B_c and C_c are canonical matrices [8].

7.1 The Tracking Problem

We want to design a control law such that the output of the system $y(t)$ asymptotically tracks a reference signal $r(t)$ i.e.

$$\lim_{t \to \infty} y(t) = r(t) \tag{7.2}$$

In order to achieve global stabilization/tracking, the first equation of (7.1), with x as the input, is assumed to be input-to-state stable. On the other hand, local stabilization only requires $\dot{z} = f_o\left(z, 0\right)$ to be asymptotically stable. Assuming that the first ρ derivatives of $r(t)$ exist and are bounded, define the reference vector and error coordinates as

$$\mathbf{r}(t) = \begin{bmatrix} r \\ \vdots \\ r^{(\rho-1)} \end{bmatrix}, \quad \xi(t) = \begin{bmatrix} x_1 - r \\ \vdots \\ x_\rho - r^{(\rho-1)} \end{bmatrix} = x(t) - \mathbf{r}(t) \tag{7.3}$$

Substituting in (7.1), we get

$$\begin{aligned} \dot{z} &= f_o\left(z, \xi + \mathbf{r}\right) \\ \dot{\xi} &= A_c \xi + B_c \left\{ \frac{1}{\beta(z,x)} \left[u - \alpha(z,x) \right] - r^{(\rho)} \right\} \\ y &= C_c x = C_c(\xi + \mathbf{r}) \end{aligned} \tag{7.4}$$

The linearizing state feedback control

$$u = \alpha(z,x) + \beta(z,x)\left[v + r^{(\rho)}\right] \tag{7.5}$$

reduces (7.1) to

$$\begin{aligned} \dot{z} &= f_o\left(z, \xi + \mathbf{r}\right) \\ \dot{\xi} &= A_c \xi + B_c v \end{aligned} \tag{7.6}$$

Asymptotic tracking is achieved by taking $v = K\xi$ such that $A_c + B_c K$ is Hurwitz.

7.2 A Special Case

If the system is full state feedback linearizable i.e. $\rho = n$ then the solution of the output feedback control problem is simple. In this case (7.1) reduces to

$$\begin{aligned} \dot{\xi} &= A_c \xi + B_c \left\{ \frac{1}{\beta(x)}\left[u - \alpha(x)\right] - r^{(n)} \right\} \\ y &= C_c x = C_c(\xi + \mathbf{r}) \end{aligned} \tag{7.7}$$

7.3 High-Gain Observers

Using the measurements from the output *y*, ξ can be estimated by high-gain observers. A formal discussion of the problem can be found in [8]. For the special case of previous section, we replace the states *x* (or ξ) with their estimates $\hat{x}$ (or $\hat{\xi}$) in (7.5) to get

$$\begin{aligned} u &= \alpha(\hat{x}) + \frac{1}{\gamma(\hat{x})}\left[v + r^{(n)}\right] \\ v &= K\hat{\xi} \end{aligned} \tag{7.8}$$

Controller (7.8) solely depends on the output of the plant and hence is of great practical significance.

8 SAMPLED-DATA CONTROL

We now design a controller for the system (7.1). The system is full-state feedback linearizable if the relative degree ρ is equal to the order of the system *n*. Continuous time controller of this system is discussed in Section 7.2. We begin the design of discrete output feedback controller by the following figure

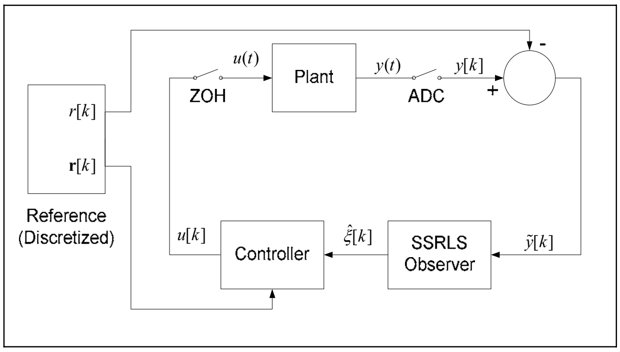

Figure 1. Sampled-Data Control of a Nonlinear System

8.1 State-Observation using High-Gain SSRLS

Rewrite (7.7) as

$$\begin{aligned} \dot{\xi}(t) &= A_c \xi(t) + B_c s(t) \\ \tilde{y}(t) &= C_c \xi(t) \end{aligned} \tag{8.1}$$

where

$$\begin{aligned} s(t) &= \frac{1}{\beta(x(t))}\left[u(t) - \alpha(x(t))\right] - r^{(n)}(t) \\ \tilde{y}(t) &= C_c\left(y(t) - r(t)\right) \end{aligned} \tag{8.2}$$

The system (8.1) can be viewed as a linear system with $\xi(t)$ as the states, $s(t)$ as a deterministic disturbance and $\tilde{y}(t)$ as the output. We now discretize (8.1) by using the modified triangular hold equivalent [12] to get

$$\begin{aligned} \xi[k+1] &= A\xi[k] + Bs[k] + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} \delta_d[k] \\ \tilde{y}[k] &= C\xi[k] \end{aligned} \tag{8.3}$$

Comparing this result with (6.1)

$$\begin{aligned} A &= \exp(A_c T) \\ C &= C_c \\ w[k] &= Bs[k] + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} \delta_d[k] \end{aligned} \tag{8.4}$$

Since $w[k]$ appears as a disturbance, smaller λ will tend to produce accurate results according to Section 6.1. The state estimates $\hat{x}[k]$ can be found from estimates of error coordinates by

$$\hat{x}[k] = \hat{\xi}[k] + \mathbf{r}[k] \tag{8.5}$$

where $\mathbf{r}[k]$ is the sampled reference vector.

8.2 Discrete Linearizing Feedback

Assume that we apply the following discrete control $u[k]$ after ZOH to the system (7.1).

$$u[k] = \alpha_o(\hat{x}[k]) + \beta_o(\hat{x}[k])\left[v[k] + r^{(n)}[k]\right] \tag{8.6}$$

where α_o and β_o are nominal models for α and β. ZOH gives us

$$u(t) = u[k] \quad kT \le t < kT + T \tag{8.7}$$

where *T* is the sampling time. (8.6) reduces (7.1) to

$$\dot{\xi} = A_c \xi + B_c\left(v[k] + \delta(t)\right) \tag{8.8}$$

where

$$\begin{aligned} \delta(t) &= \frac{1}{\beta(x)}\left\{\delta_1(t) + \delta_2(t)\left(v[k] + r^{(n)}[k]\right)\right\} + \delta_3(t) \\ \delta_1(t) &= \alpha_o\left(\hat{x}[k]\right) - \alpha\left(x(t)\right) \\ \delta_2(t) &= \beta_o\left(\hat{x}[k]\right) - \beta\left(x(t)\right) \\ \delta_3(t) &= r^{(n)}[k] - r^{(n)}(t) \end{aligned} \tag{8.9}$$

We have incorporated estimation error, discretization error and model uncertainty in (8.9)

8.3 Stabilizing Control

Integrate (8.8) for time interval $\left[kT, kT+T\right]$

$$\begin{aligned} \xi[k+1] = {} & \exp(A_c T)\xi[k] + \\ & \int_{kT}^{kT+T} \exp\left(A_c(kT + T - \tau)\right) B_c\left(v[k] + \delta(\tau)\right) d\tau \end{aligned} \tag{8.10}$$

which may be simplified into

$$\xi[k+1] = A_o\xi[k] + B_o v[k] + \Delta_1[k] \tag{8.11}$$

by introducing $A_o, B_o, \Delta_1[k]$ [12]. Now we design a discrete control $v[k] = K_o\hat{\xi}[K]$ such that $A_o + B_o K_o$ has all its eigenvalues within the unit circle. This transforms second equation of (8.10) into

$$\xi[k+1] = (A_o + B_o K_o)\xi[k] + \Delta[k] \tag{8.12}$$

where $\Delta[k]$ incorporates the effect of error in estimation of $\xi[k]$. By Theorem 6.1

$$\lim_{k\to\infty} \|\xi[k]\| \le \frac{\alpha}{1-\rho(A_o + B_o K_o)} \sup_{k\ge 0} \|\Delta[k]\| \tag{8.13}$$

for some positive constant α. This bound shows that high-gain control has disturbance rejection property.

8.4 Integral Control

In many practical problems, $\delta(t)$ of (8.8) settles down asymptotically to a value that contains a DC component. Suppose

$$\lim_{t\to\infty} \delta(t) = \delta_o + \delta_{ac}(t) \tag{8.14}$$

The integral control has the ability of rejecting δ_o asymptotically.

9 EXAMPLE (INVERTED PENDULUM)

Consider the following pendulum equation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -10\sin(x_1) - x_2 + 1.2\,u \\ y &= x_1 \end{aligned} \tag{8.15}$$

We want the output $y(t)$ to track a reference $r(t) = 2 + \sin(t)$. The system (8.15) is already in normal form. The nominal models used in the linearizing feedback control (Section 8.2) are taken as

$$\begin{aligned} \alpha_o(x[k]) &= 11\sin(x_1[k]) + x_2[k] \\ \beta_o(x[k]) &= 1 \end{aligned} \tag{8.16}$$

We use SSRLS with forgetting factor 0.75. The results are illustrated in Figure 2. The controller of Section 8.3 provides good transient behavior at the cost of larger steady state tracking error. On the other hand Integral controller exhibits superior tracking performance with deterioration in transient behavior.

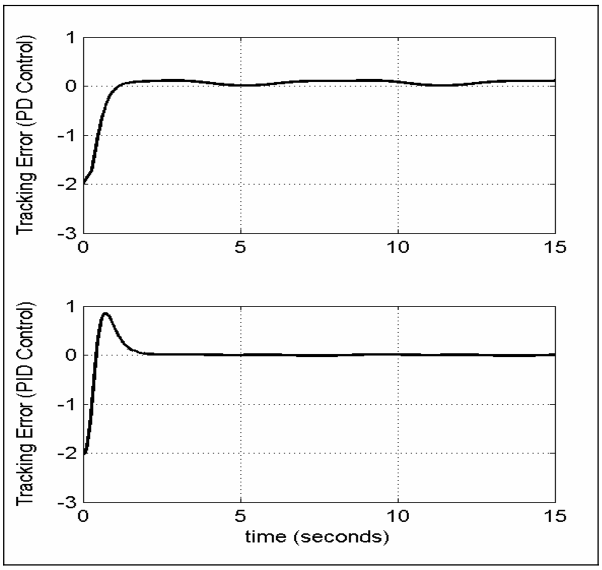

Figure 2. Comparison of Tracking Performances

10 CONCLUSIONS

The use of SSRLS in nonlinear control systems is a major breakthrough in the sense that it formalizes the application of adaptive filters as state estimators in sampled-data control systems. Previously designers have not been able to accomplish this combination as the relevance between adaptive filters and state estimators has not been this straightforward. The analogy between continuous high-gain observers and high-gain SSRLS is expected to produce a significant impact on design and applications related to sample-data control systems.

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