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On the steady state behaviour of a class of non-linear mappings.

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Abstract– It is shown that a semi-invariant and globally attracting belt exists for a particular class of non-linear mappings. A Sigma-Delta modulator with a quasiperiodic input with integrator leakage is an example of an electronic system described by such a mapping.

1. Introduction

In previous work [1-3] it was established that a specific class of nonlinear mappings possesses the notable property that in steady state there exists a semi-invariant globally attracting *belt*. A belt is a set bounded by two contours. A set is semi–invariant with respect to a mapping if its image under the mapping is contained within itself [4-6]. A set is globally attracting if every orbit converges to this set. These type of mappings arise in the models of a number of electronic circuits that incorporate quantisation e.g. Sigma-Delta modulators and Digital Phase Locked Loops. In [2], the application of these mappings to models of Sigma-Delta modulators and Digital Phase Locked Loops is discussed in detail. In this paper, it is shown that this result, namely the existence of a semi-invariant and globally attracting belt, holds for a more general class of non-linear mappings. In section 3, the new theorems are presented. In section 4, the results are applied to the case of a multibit Sigma-Delta modulator with a quasiperiodic input with integrator leakage [6-8].

2. Previous results

In [1-3] the non-linear mapping considered is of the following form:

$$M:\begin{pmatrix}\theta\\ u\end{pmatrix}\rightarrow\begin{pmatrix}(\theta+\omega)\quad \mathrm{mod}(2\pi)\\ u-g(\theta+\omega)sign(u)+f(\theta+\omega)\end{pmatrix}\tag{1}$$

where $f(\theta)$ and $g(\theta)$ are continuous periodic functions of period 2π and θ is an angular variable where $\theta\in[0,2\pi]$.

The sign function is defined as follows:

$$sign(u)=\begin{cases}1 & u\ge 0\\ -1 & u<0\end{cases}\tag{2}$$

In [1-3] it is shown that if $g(\theta)>0$ for every $\theta\in[0,2\pi]$, and the magnitude of the average of $f(\theta)$ is less than the average of $g(\theta)$, then there exists a belt

$$B=\{(\theta,u)\ |L(\theta)\le u\le U(\theta)\}$$

such that B is semi-invariant, i.e. $M(B)\subset B$,the bounding contours U and L are continuous, and B is globally attracting. In fact, if $|f(\theta)|<g(\theta)$ for all θ, then the boundaries of the belt are given by: $U(\theta)=f(\theta)+g(\theta),\ L(\theta)=f(\theta)-g(\theta)$ for all $\theta\in[0,2\pi]$.

These results can be applied to the ideal first-order single bit Sigma-Delta modulator and a first-order bang-bang phase-locked loop as discussed in [2-3].

In section 3, it is shown how these results namely the existence of a semi-invariant and globally attracting belt, holds for a more general class of non-linear mappings.

3. Main Theorem

3.1 The non-linear mapping.

Consider the following non-linear mapping:

$$M:\begin{pmatrix}\theta\\ u\end{pmatrix}\rightarrow\begin{pmatrix}(\theta+\omega)\quad \mathrm{mod}(2\pi)\\ F(u,g(\theta+\omega))+f(\theta+\omega)\end{pmatrix}\tag{3}$$

where $f(\theta)$, $g(\theta)$ are continuous periodic functions of period 2π.

In order to show that a semi-invariant and globally attracting belt exists for the mapping in (3), two lemmas are required. The first lemma defines $F(u,g(\theta))$ and describes the boundaries of $F(u,g(\theta))$. The second lemma is concerned with the bounding contours $U(\theta)$ and $L(\theta)$ of the belt.

Lemma 1: *Let $F(u,g(\theta))$ be an odd function in u satisfying*:

$$F(u, g(\theta)) = pu + cg(\theta) \qquad u \le \left(\frac{k-c}{p}\right) g(\theta)$$

$$-kg(\theta) < F(u, g(\theta)) < kg(\theta) \quad \left(\frac{k-c}{p}\right) g(\theta) < u < \left(\frac{k+c}{p}\right) g(\theta)$$

$$F(u, g(\theta)) = pu - cg(\theta) \qquad u \ge \left(\frac{k+c}{p}\right) g(\theta)$$

where $0 < p < 1, c > 0$ *and* $k > 0$ *(c.f. Figure 1)* *(A.1)*

then
there exist an upper bound and lower bound $\xi(u, g(\theta))$ $\eta(u, g(\theta))$ *of* $F(u,g(\theta))$ *such that*:

$$\eta(u,g(\theta)) = -\xi(-u,g(\theta)) \qquad (A.2)$$

$$\eta(u'',g(\theta)) < F(u,g(\theta)) < \xi(u',g(\theta)) \qquad \textit{for all } u'' < u < u' . \qquad (A.3)$$

$\xi(u,g(\theta))$ *and* $\eta(u, g(\theta))$ *are continuous, monotone non-decreasing in* u *,* *(A.4)*

$$\xi(u + \sigma,g(\theta)) \le \xi(u,g(\theta)) + p\sigma \text{ and}$$
$$\eta(u - \sigma,g(\theta)) \ge \eta(u,g(\theta)) - p\sigma$$
$$\textit{for all } \sigma \ge 0 \textit{, for all } u, \qquad (A.5)$$

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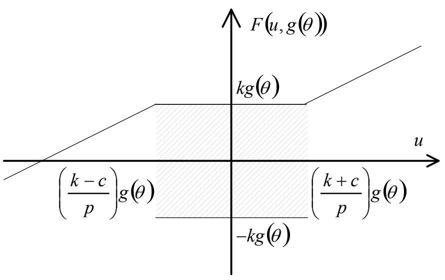

Figure 1 : The region in which the graph of $F(u,g(\theta))$ can lie.

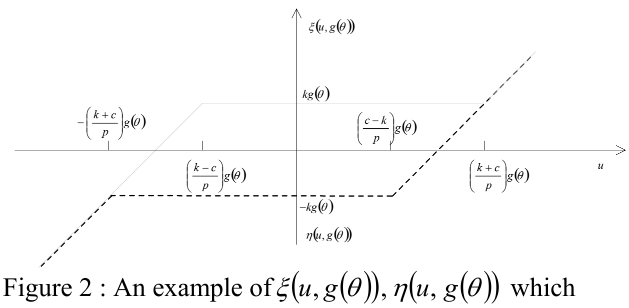

Figure 2 : An example of $\xi(u, g(\theta)), \eta(u, g(\theta))$ which satisfy (A.2-A.5).

The functions $\xi(u, g(\theta)), \eta(u, g(\theta))$ relate to the bounding contours of the belt $U(\theta)$ and $L(\theta)$.

Lemma 2: *If* $g(\theta) > 0$ *for all* θ*, then for any upper and lower bound* $\xi(u, g(\theta)), \eta(u, g(\theta))$ *which satisfy (A.2-A.6) there exist unique functions* $U(\theta)$ *and* $L(\theta)$ *defined on* $[0, 2\pi]$ *such that:*

$$U(\theta) = f(\theta) + \xi(U(T^{-1}\theta), g(\theta))$$
$$L(\theta) = f(\theta) + \eta(L(T^{-1}\theta), g(\theta)) \qquad (B.1)$$

where $T\theta = (\theta + \omega) \bmod(2\pi)$,

U and L are continuous, *(B.2)*

$U(\theta) \ge L(\theta)$ *for all* $\theta \in [0, 2\pi]$. *(B.3)*

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Theorem 1 presents the conditions under which the bounding contours $U(\theta)$ and $L(\theta)$ which satisfy *(B.1-B.3)* describe a belt that is semi-invariant and globally attracting.

Theorem 1: *If* $g(\theta) > 0$ *for all* θ *and if* $F(u,g(\theta))$ *satisfies the conditions of lemma1, then there exist contours* $U(\theta)$ *and* $L(\theta)$ *which satisfy (B.1-B.3) such that there exists a belt given by:*

$$B = \{(\theta,u) \mid L(\theta) \le u \le U(\theta)\} \quad \textit{for all} \quad \theta \in [0, 2\pi]$$

which is semi-invariant and globally attracting.

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The proofs of lemma 1, lemma 2 and Theorem 1 are not included due to lack of space.

Theorem 2 is an extension of Theorem 1. It shows that if the input $f(\theta)$ lies within a certain range, then the bounding contours $U(\theta)$ and $L(\theta)$ have a simple form.

Theorem 2: *If* $g(\theta) = g > 0$ *for all* θ*, if* $F(u,g(\theta))$ *satisfies the conditions of lemma1 and if*

$$|f(\theta)| < g\left[\left(\frac{k+c}{p}\right)(1-p) + c\right] \textit{ for all } \theta \in [0, 2\pi], \textit{ then}$$

there exists a $U(\theta)$ *and* $L(\theta)$ *which satisfy (B.1-B.3) and are given as:*

$$U(\theta) = f(\theta) + kg(\theta)$$
$$L(\theta) = f(\theta) - kg(\theta) \qquad \textit{for all} \quad \theta \in [0, 2\pi].$$

These $U(\theta)$ *and* $L(\theta)$ *describe a belt B,*

$$B = \{(\theta,u) \mid L(\theta) \le u \le U(\theta)\} \textit{ for all } \theta \in [0, 2\pi] \qquad (C.1)$$

which is semi-invariant and globally attracting.

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Proof of Theorem 2:
Define $\xi(u, g(\theta))$ as follows:

$$\xi(u,g(\theta))=\begin{cases} pu+cg(\theta) & u\le -\left(\frac{k+c}{p}\right)g(\theta) \\ kg(\theta) & -\left(\frac{k+c}{p}\right)g(\theta)<u<\left(\frac{k+c}{p}\right)g(\theta) \\ pu-cg(\theta) & u\ge \left(\frac{k+c}{p}\right)g(\theta) \end{cases} \qquad (D.1)$$

and define $\eta(u, g(\theta))$ so that it satisfies (A.2). It can be shown that $\xi(u, g(\theta))$, $\eta(u, g(\theta))$ satisfy (A.2-A.6). It follows that the conditions of lemma 2 hold. Hence, there exist unique functions $U(\theta)$ and $L(\theta)$ which satisfy *(B.1-B.3)*.

Furthermore, consider

$$\begin{aligned}\tilde{U}(\theta)&=f(\theta)+kg(\theta)\\ \tilde{L}(\theta)&=f(\theta)-kg(\theta)\end{aligned} \quad \text{for all } \theta\in[0, 2\pi].$$

By *(D.1)* $\xi\left(f\left(T^{-1}\theta\right)+kg\left(T^{-1}\theta\right), g(\theta)\right)=kg(\theta)$

If $\left|U\left(T^{-1}\theta\right)\right|\le\left(\frac{k+c}{p}\right)g(\theta)$ then

$$\left|f\left(T^{-1}\theta\right)+kg\left(T^{-1}\theta\right)\right|\le\left(\frac{k+c}{p}\right)g(\theta)$$

$$\equiv\left|f\left(T^{-1}\theta\right)\right|\le g\left[\left(\frac{k+c}{p}\right)(1-p)+c\right]$$

as $g(\theta)=g\left(T^{-1}\theta\right)=g$.

Since, $|f(\theta)|<g\left[\left(\frac{k+c}{p}\right)(1-p)+c\right]$ for all $\theta\in[0, 2\pi]$,

then

$$\tilde{U}(\theta)=f(\theta)+\xi\left(\tilde{U}\left(T^{-1}\theta\right), g(\theta)\right)=f(\theta)+kg(\theta)$$

Similarly, for $\tilde{L}(\theta)$.

$\tilde{U}(\theta)$, $\tilde{L}(\theta)$ satisfy *(B.1-B.3)*.

These functions are unique and hence,

$$\begin{aligned}U(\theta)&=\tilde{U}(\theta)=f(\theta)+kg(\theta)\\ L(\theta)&=\tilde{L}(\theta)=f(\theta)-kg(\theta)\end{aligned} \quad \text{for all } \theta\in[0, 2\pi].$$

By Theorem 1, the contours $U(\theta)$ and $L(\theta)$ describe a belt given by:

$$B=\{(\theta,u)\ |L(\theta)\le u\le U(\theta)\} \quad \text{for all} \quad \theta\in[0, 2\pi]$$

which is semi-invariant and globally attracting.

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4. Sigma-Delta Modulator

Consider the multibit Sigma-Delta (ΣΔ) modulator with integrator leakage as shown in Figure 3. ΣΔ modulators are used in analog-to-digital and digital-to-analog conversion and are based around the use of feedback to improve the effective resolution of a coarse quantiser [7-8].

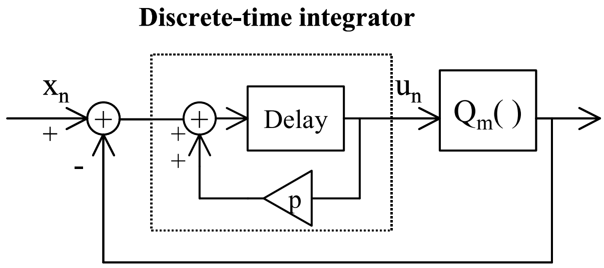

Figure 3 : Discrete-time first-order ΣΔ modulator.

Figure 3 shows the basic first order ΣΔ modulator in discrete time with integrator leakage [7] incorporated into the ΣΔ model by taking $0 < p < 1$ in the system of Figure 3. The non-ideal discrete-time system is modelled by the equation

$$u_{n+1}=pu_n+x_{n+1}-Q_m(u_n) \qquad (4)$$

where

$$Q_m(u)=\begin{cases} \frac{1}{m-1} & 0\le u\le\frac{2}{m-1} \\ \frac{3}{m-1} & \frac{2}{m-1}\le u\le\frac{4}{m-1} \\ \frac{5}{m-1} & \frac{4}{m-1}\le u\le\frac{6}{m-1} \\ \vdots & \\ 1 & \frac{m-2}{m-1}\le u \\ -\frac{1}{m-1} & -\frac{2}{m-1}\le u\le 0 \\ \vdots & \\ -1 & u\le-\frac{m-2}{m-1} \end{cases} \qquad (5)$$

Let $x_n=f(\theta_n)$ where $\theta_{n+1}=(\theta_n+\omega)\bmod(2\pi)$ and $f(\theta)$ is a continuous periodic function with period 2π and θ is an angular variable where $\theta\in[0, 2\pi]$.

Converting equation (4) to autonomous form:

$$\begin{bmatrix}\theta_{n+1}\\ u_{n+1}\end{bmatrix}=\begin{bmatrix}(\theta_n+\omega)\bmod(2\pi)\\ pu_n-Q_m(u_n)+f(\theta_n+\omega)\end{bmatrix}\equiv M\left(\begin{bmatrix}\theta_n\\ u_n\end{bmatrix}\right) \qquad (6)$$

where M is given as follows with $g(\theta)\equiv 1$:

$$M:\begin{pmatrix}\theta\\ u\end{pmatrix}\rightarrow\begin{pmatrix}\theta+\omega \quad \bmod(2\pi)\\ pu\text{-}g(\theta+\omega)Q_m(u)+f(\theta+\omega)\end{pmatrix} \qquad (7)$$

This is in the form of equation (3), with

$$F(u,g(\theta))=pu-g(\theta)Q_m(u),\ g(\theta)\equiv 1. \qquad (8)$$

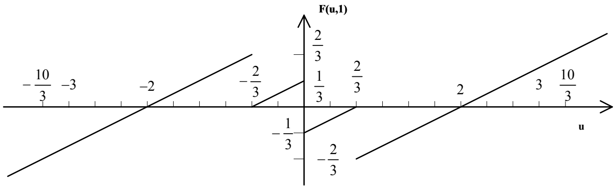

Figure 4 : Graph of $F(u,1)$ with m=4 and p=0.5.

Consider the following example. Figure 4 shows the graph of $F(u,g(\theta))$ where $F(u,1)$ with $p=0.5$ and $m=4$. Comparing Figure 4 with Figure1, it is clear that $F(u,g(\theta))$ satisfies the conditions of lemma 1 with $p=0.5$, $c=1$ and $k=2/3$. Since, $g(\theta)\equiv 1$ and if $|f(\theta)|<g(\theta)\left[\left(\frac{k+c}{p}\right)(1-p)+1\right]=\frac{8}{3}$ then the conditions of Theorem 2 are satisfied. Hence, $U(\theta)=f(\theta)+\frac{2}{3}$ and $L(\theta)=f(\theta)-\frac{2}{3}$ are the bounding contours of a belt which is semi-invariant and globally attracting with respect to system (6).

Figure 5 shows that when $|f(\theta)|<\frac{8}{3}$ for all $\theta\in[0,2\pi]$ the orbit is bounded by the belt given in (E.1). Figure 6 shows that when $|f(\theta)|>\frac{8}{3}$ for some $\theta\in[0,2\pi]$ the orbit is no longer bounded by the belt given in (E.1).

5. Conclusion

It has been shown that for a general class of non-linear mappings, there exists a semi-invariant, globally attracting belt. In fact, if the input lies within a certain range, the boundaries of the belt have a simple form. These results are applied to the case of a multibit Sigma-Delta modulator with integrator leakage.

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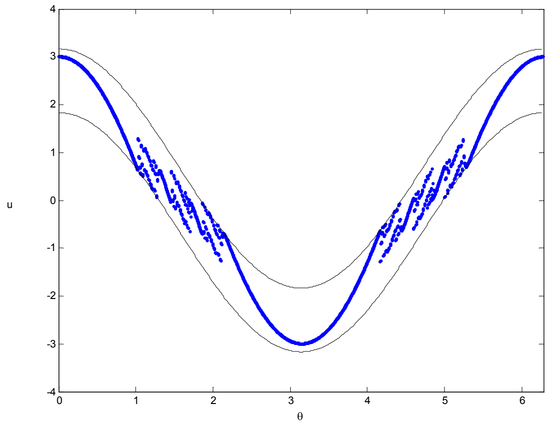

Figure 5 : An orbit of (5) with with $f(\theta)=2.5\cos(\theta)$, $\omega=0.01$, m=4, p=0.5. The belt bounded by $2.5\cos(\theta)+2/3$ and $2.5\cos(\theta)-2/3$ is plotted.

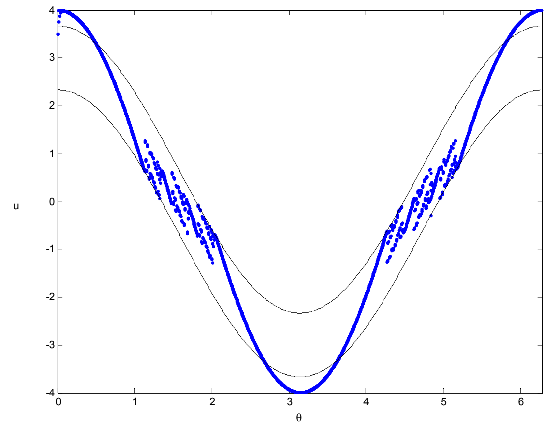

Figure 6 : An orbit of (5) with with $f(\theta)=3\cos(\theta)$, $\omega=0.01$, m=4,p=0.5. The belt bounded by $3\cos(\theta)+2/3$ and $3\cos(\theta)-2/3$ is also plotted.