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Properties of Level Crossing Intervals of the Chaotic Process Generated by the Logistic Map

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Abstract– Level-crossing statistics of random processes have been intensively studied since the pioneering work of S.O. Rice [1]. In this paper we extend the level crossing analysis to chaotic time series. As a basic vehicle we use the logistic map as process generator. In analogy to the approach for random processes we analyze the statistical characteristics mean, variance, probability distribution of the level crossing intervals, and correlation properties between adjacent level crossing intervals both theoretically and experimentally. We present some interesting features of level crossing intervals. We also demonstrate that these characteristics are typical for chaotic processes and widely different from those of stationary random process such as Gaussian processes..

1. Introduction

Since the pioneering work of S.O. Rice [1] many authors have analyzed and discussed the statistical properties of level crossings and level crossing intervals for stationary random processes such as Gaussian or Rayleigh processes. In this paper we analyze the statistical properties of level crossing intervals of the chaotic process generated by the logistic map in eq. (2) and depicted in Fig.l. The results of this paper can be extended to the class of chaotic sequences which are generated by other chaotic maps equivalent to the logistic map, for example the Tent map.

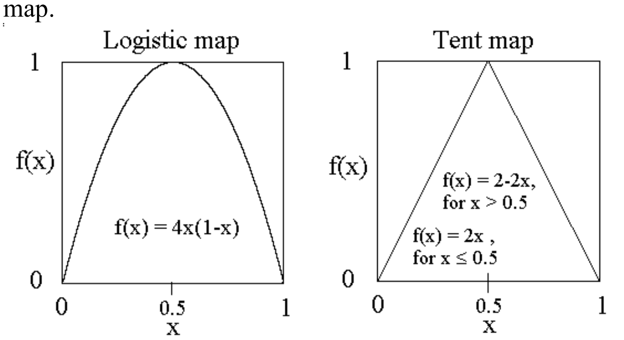

Fig. 1 Map functions

As shown in Fig. 2 the chaotic sequence is converted into a binary signal by comparing every signal value with the level value L. The time moment where the discrete signal first-time exceeds the crossing level is called crossing point. Depending on the direction of crossing we distinguish up-crossing and down-crossing. The intervals between the edges of the binary signal are an integer multiple of the unit interval T between the discrete signal samples. T is set to 1 throughout this paper. The statistical properties of mean, variance, probability distribution of the crossing intervals, and the correlation between the crossing intervals are studied both theoretically and experimentally.

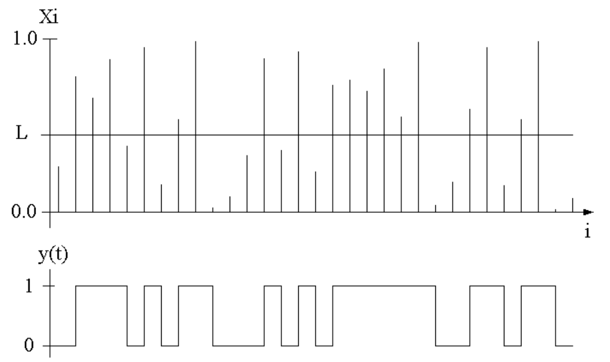

Fig.2 Generation of square-wave signal

2. Basic Definitions

Let X_i , i = 0,1,... be the discrete process values generated by a return map

$$X_i = F(X_{i-1}). \quad (1)$$

For the logistic map F is given by

$$F(x) = 4x(1-x), \quad (2)$$

and thus the sequence X_i is defined by

$$X_i = 4X_{i-1}(1-X_{i-1}), \quad \text{for } 0 < X_0 < 1 \text{ and } i > 0. \quad (3)$$

As illustrated in Fig.2, the continuous-time binary or square-wave signal $y(t)$ is generated from the discrete random process X_i by comparison with the level L as follows:

$$\begin{aligned} y(t) &= 1, && \text{for } X_i > L \text{ and } t = iT, \\ &= 0, && \text{for } X_i \le L \text{ and } t = iT, \quad (4) \\ &= y(iT), && \text{for } iT < t < (i+1)T. \end{aligned}$$

where T is the length of the time interval between the data points i and i+1 of the discrete-time process X_i.
For the time intervals between crossings we introduce the following notations:

τ_{n+} is the length of an interval between an up-crossing as first crossing and the n+2 th crossing.
τ_{n-} is the length of an interval between a down crossing as first crossing and the n+2 th crossing.
The index n means the number of crossings inside the interval, and +- are the polarity of the start crossing.
Fig.3 illustrates these definitions.

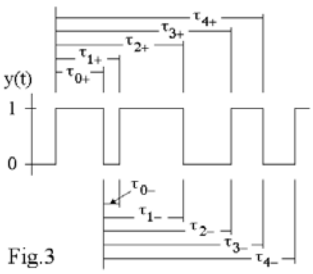

Fig.3
Definition of crossing intervals

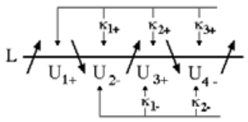

Fig.4
Definition of correlations between crossing intervals

For these level crossing intervals the probability distributions are defined as follows:
$p_{n+}(k) = P(\tau_{n+} = k)$, i.e. the probability that the crossing interval τ_{n+} has the length k, and
$p_{n-}(k) = P(\tau_{n-} = k)$, i.e. the probability that the crossing interval τ_{n-} has the length k.
The mean values and variances of the crossing intervals are denoted as follows:
$\mu_{n+} = E(\tau_{n+})$ and $\mu_{n-} = E(\tau_{n-})$ are the mean values of the above interval lengths respectively and
$\sigma^2_{n+} = E(\tau_{n+} - \mu_{n+})^2$ and $\sigma^2_{n-} = E(\tau_{n-} - \mu_{n-})^2$ are the variance values of τ_{n+} and τ_{n-} respectively.
As illustrated in Fig. 4, the correlation sequences between successive intervals U_i are defined as follows;

$$\kappa_{i\pm} = \left[E\{U_{n\pm}U_{n+i,\mp}\} - \mu_{0+}\mu_{0-}\right]/(\ _{0+}\ _{0-}), \quad \text{for i=1,3,5,.. ,} \quad (5)$$

$$\kappa_{i\pm} = \left[E\{U_{n\pm}U_{n+i,\pm}\} - \mu_{0\pm}^2\right]/\ ^2_{0\pm}, \quad \text{for i=2,4,6,.. ,} \quad (6)$$

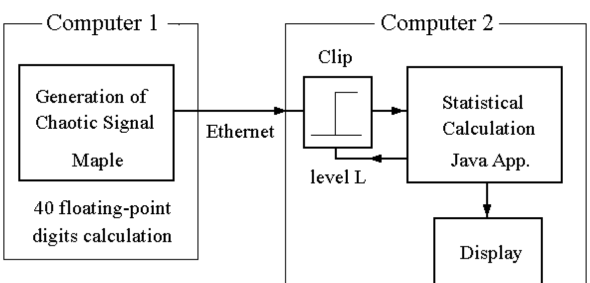

Fig. 5 Experimental setup

3. Experimental Setup

As shown in Fig. 5, the experimental setup consists of two functional blocks. One block is for a chaotic signal generation, and the other block is for measurement of level crossing intervals and statistical calculations for these intervals. These two blocks are connected via Ethernet. The calculation of chaotic signal generation is done by Maple with 40 decimal digits of floating-point calculation. The calculated results are written on data files and they are transferred to the measurement block. In the measurement block the chaotic signal is clipped at level L in order to produce crossing intervals, and for these intervals the crossing rates, probability distributions, mean values, variances, and correlation of the intervals are measured.

4. Experimental and theoretical results

The probability distributions $p_{0+}(k)$, $p_{0-}(k)$, $p_{1+}(k)$, $p_{1-}(k)$ of the level crossing intervals have been measured for different levels between 0 and 1. On the other hand the exact solutions for the probability distributions p_0 and p_1 have been derived in the form of recursive expressions and also partially in a closed form. Because of the not enough space we mention here only for the derivations of $p_{0+}(k)$ and $p_{1+}(k)$. For the derivation of $p_{0+}(k)$, the following functions are defined.

$$f(x) = 4x(1\quad x), \qquad (7)$$

$$\begin{cases} g(0,x) = f(x), & n = 0 \\ g(n,x) = f^{(n+1)}(x) = f(g(n\quad 1,x)), & n \geq 1 \end{cases} \qquad (8)$$

$$g_{cr}(n,x) \equiv \begin{cases} 1 & g(n,x) > L, \quad \text{for} \quad n \geq 0, \\ 0 & g(n,x) \leq L, \quad \text{for} \quad n \geq 0, \end{cases} \qquad (9)$$

$$ng_{cr}(n,x) \equiv \begin{cases} 0 & g(n,x) > L, \quad \text{for} \quad n \geq 0, \\ 1 & g(n,x) \leq L, \quad \text{for} \quad n \geq 0. \end{cases} \qquad (10)$$

where $g(n,x)$ is the $(n+1)$th iterate of $f(x)$, and $g_{cr}(n,x)$ is the clipped function of $g(n,x)$ by level L. The value 1 or 0 of $g_{cr}(n,x)$ indicates, whether the process, which started at initial value x, stays above or beneath the level L at step n. The function $ng_{cr}(n,x)$ is the complement of $g_{cr}(n,x)$. Now we determine the region or set of the initial values x of a process, which cross the level L upward at the step n = 1. This region R_1 can be written as

$$R_1 = [f_1(L), min(f_2(L), L)], \qquad \text{where}$$

$$f_1(x) = (1 - \sqrt{1-x})/2, \text{ and } f_2(x) = (1 + \sqrt{1-x})/2. \qquad (11)$$

$$\begin{cases} h(0,x) \equiv f_{ar}(x) \equiv \begin{cases} 1 & f_1(L) \leq x < min(f_2(L), L), \\ 0 & \text{others}, \end{cases} \\ h(n,x) \equiv g_{cr}(n-1,x) \bullet h(n-1,x), \quad for\ n \geq 1. \end{cases} \qquad (12)$$

The function $h(0,x)$ in (12) shows the region of initial value x of the process, which cross the level L upward as the start up-crossing at step n = 1. The function $h(n,x)$ shows the region of the initial value x, that the process remains above the level L without downcrossing L until step n. The function $h(n+1,x)$ can be calculated as the cross section of the region of $h(n,x)$ and $g_{cr}(n,x)$ in a recursive form. The development of the functions is illustrated in Fig. 6

The probability $S(n)$ that the process remains above the level L until step n, can be calculated by integration of the probability density $f_p(x)$ for the region $h(n,x)$ as

$$S(n) = \int_0^1 h(n,x) f_p(x)dx, \quad \text{for } n \geq 0, \qquad \text{where} \quad f_p = 1\Big/\left(\pi\sqrt{x(1-x)}\right). \tag{13}$$

The probability distribution $P_{0+}(k)$ of the level crossing intervals is calculated from the difference of the probability $S(n)$ at step k and $k+1$ as;

$$p_{0+}(k) = \{S(k) - S(k+1)\}/S(0). \tag{14}$$

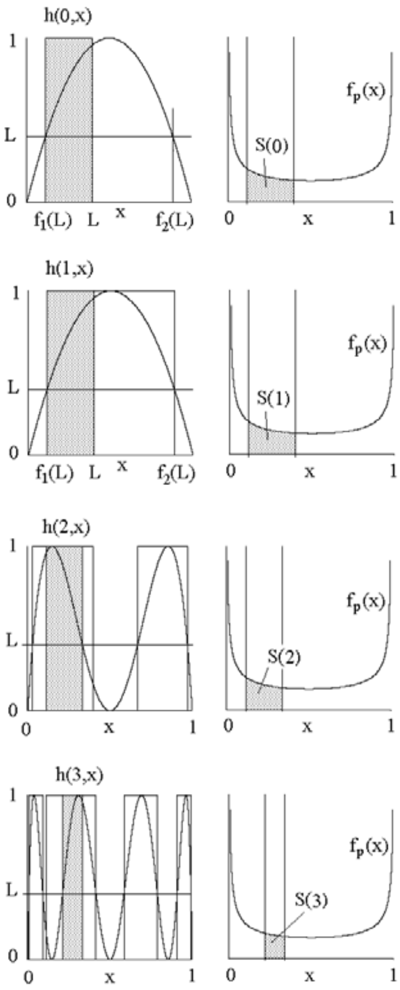

Fig. 6 Development of the functions g(n,x), h(n,x) and S(n)

For the derivation of $p_{1+}(k)$ the following functions are defined and calculated in a recursive way.

$$h_{s2}(0,0,x) \equiv 0, \quad \text{for } n = 0, \tag{15}$$

$$h_{s2}(0,1,x) \equiv 0, \quad \text{for } n = 0, \tag{16}$$

$$h_{s2}(n,0,x) \equiv n g_{cr}(n-1,x) h_{s2}(n-1,1,x), \quad \text{for } n \geq 1, \tag{17}$$

$$h_{s2}(n,1,x) \equiv cr(h_{s2}(n,0,x) + n g_{cr}(n-1,x) h(n-1,x)), \quad \text{for } n \geq 1, \tag{18}$$

$$cr(x) \equiv \begin{cases} 1 & x > 0, \\ 0 & x \leq 0. \end{cases} \tag{19}$$

$$S_2(n,i) = \int_0^1 h_{s2}(n,i,x) f_p(x)dx, \quad \text{for } n \geq 0, i = 0,1. \tag{20}$$

$$p_{1+}(k) = \{S_2(k,1) - S_2(k+1,0)\}/S(0). \tag{21}$$

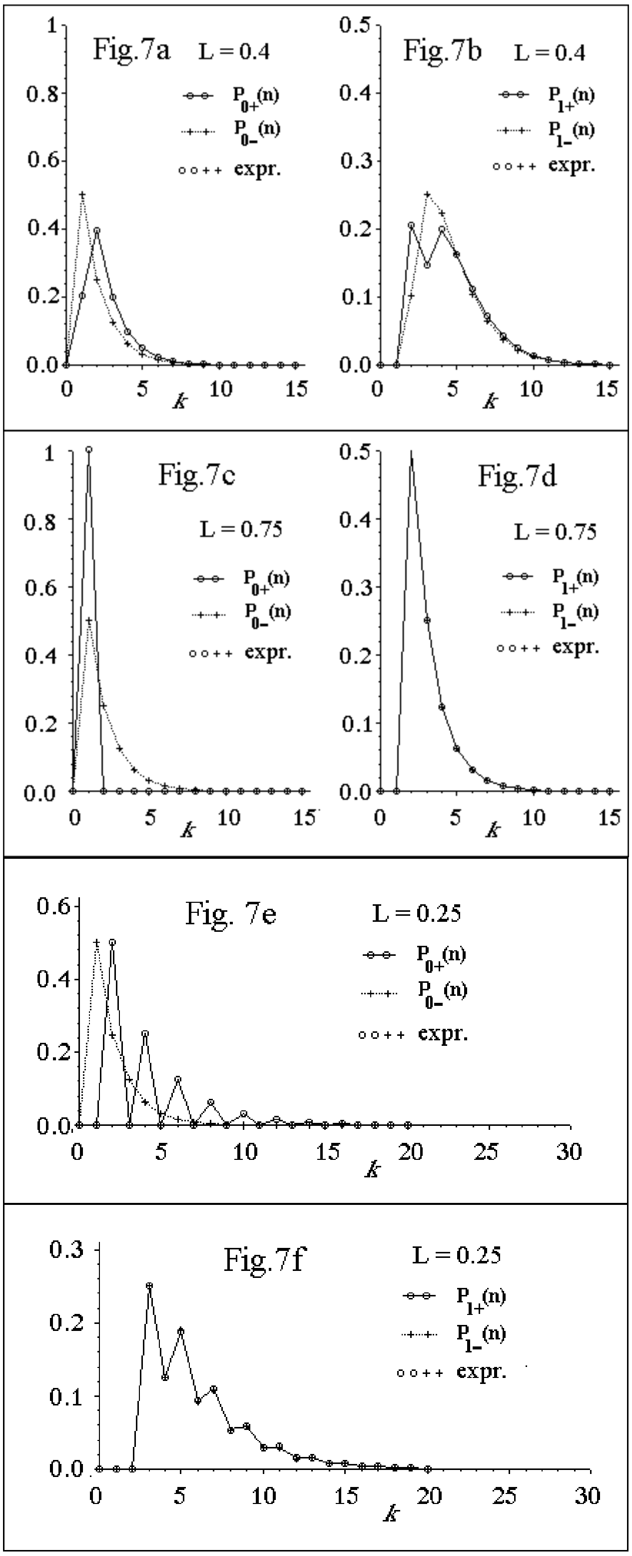

Fig.7a-7f Theoretical and experimental results of probability distributions $p_0(k)$ and $p_1(k)$

The function $h_{s2}(n,0,x)$ shows the region of the initial value x, that the process remains below the level L until step n, after having the 1st downcrossing before step n. In the same manner the function $h_{s2}(n,1,x)$ shows the region of the initial value x, that the process remains below the level L until step n, after having the 1st downcrossing until step n (including the 1st downcrossing at step n).

Some typical results for the probability distributions p_0 and p_1 are shown in Fig.7. Here both theoretical and experimental value are plotted in the same diagram, and they agree very well. Some remarkable properties of the probability distributions of level crossing intervals are summarized here:

1) The Level crossing intervals of this process show completely different properties depending on the level value L. Roughly three regions ($0 < L < 0.25$, $0.25 < L < 0.75$, and $0.75 < L < 1$) can be distinguished. The border values $L = 0.25$, $L = 0.5$ and $L = 0.75$ are special values.

2) The probability distributions of $p_{1+}(k)$ and $p_{1-}(k)$ are different in general, whereas the first moments of them are equal, i.e. $\mu_{1+} = \mu_{1-}$. This property is quite different from that of Gaussian process where $p_{1+}(k)$ and $p_{1-}(k)$ are always equal. This is an interesting and remarkable result of this analysis.

3) $p_{1+}(k)$ and $p_{1-}(k)$ are equal in the region $L > 0.75$, and at the level values $L = 0.25, 0.5, 0.75$.

4) For the level $L >= 0.75$, $p_{0+}(k)$ has the value 1 at $k = 1$, and 0 for other k. Corresponding to this $p_{1+}(k)$ and $p_{1-}(k)$ are equal to the probability distribution, which is obtained by shifting $p_{0-}(k)$ right one step.

i.e. $p_{1+}(k) = p_{1-}(k) = p_{0-}(k-1)$.

5) For the level value $L \leq 0.75$, $p_{0-}(k)$ has the same distribution which and decreases exponentially with k:

$p_{0-}(k) = (0.5)^k$, and $p_{0-}(0) = 0$.

6) $p_{0+}(k)$ and $p_{0-}(k)$ are equal at level value $L = 0.5$.

7) At $L = 0.25$, every second value of $p_{0+}(k)$ is zero, i.e. $p_{0+}(k) = 0$, for odd k and $k = 0$, and

$p_{0+}(k) = (0.5)^{(n/2)}$, for even k.

5. Correlation between successive intervals

Figs. 8a and 8b show the correlation coefficients κ_{1+} and κ_{1-} in dependence on the level L, and both theoretical and experimental results are plotted in the same diagramm. The most remarkable fact is that κ_{1+} and κ_{1-} are not equal. κ_{1+} has several zero points, e.g. $\kappa_{1+} = 0$ at level values $L = 0.25$, 0.5, 0.75, and oscillates between positive and negative values in the range of $L < 0.75$. On the other hand $\kappa_{1-} = 0$ for all values of level L. This result is rather interesting, hence κ_{1+} and κ_{1-} must be equal for stationary random processes.

6. Discussion and Conclusion

The level crossing intervals of the chaotic process derived from the logistic map show a lot of interesting properties. Mostly remarkable is that the polarity dependence of the crossing intervals, i.e. the statistical characteristics of the crossing intervals like probability distributions and correlations are strongly determined by the direction of crossing, whether the intervals start with up crossing or down crossing. The properties of this chaotic process are quite different from the properties of stationary random processes like the Gaussian process. A possible application of the results shown can be in chaos generator design.

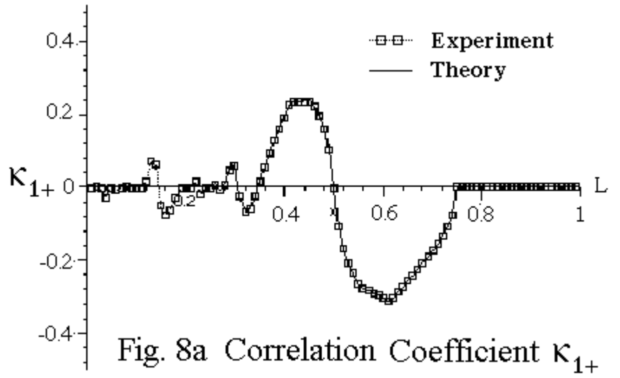

Fig. 8a Correlation Coefficient κ_{1+}

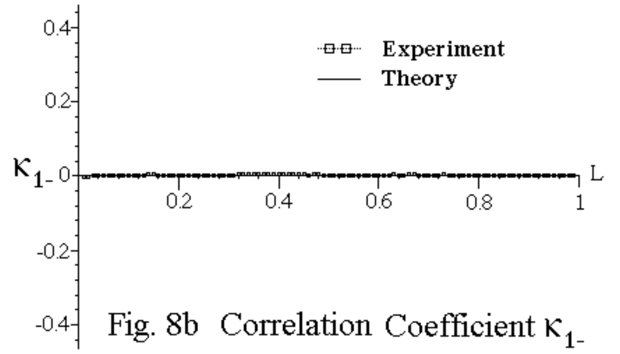

Fig. 8b Correlation Coefficient κ_{1-}

References

[1] S. O. Rice, "The Mathematical Analysis of Random Noise," *B.S.T.J.*, 24,1945, 46-156.

[2] T. Mimaki and T. Munakata, "Experimental Results on the Level-Crossing Intervals of Gaussian Processes," *IEEE.Trans. on Inf. Theory*, IT-24 1978, 515-520.

[3] T. Munakata, "Mehr - Zustaende - Modelle zur Beschreibung des Pegelkreuzungsverhaltens Stationaerer Stochastischer Prozesse," *Inaugural - Dissertation*, J. W. Goethe - Universitaet Frankfurt am Main 20.03.1986.

[4] A. Lasota and M. C. Mackey, "Chaos, Fractals and Noise," *Applied Mathematical Science,* vol. 97, Springer Verlag, 1995.

[5] M. E. Yalcyn, J. A.K. Suykens and J. Vandewalle, "A double scroll based true random generator," *ISCAS 2004*, Vancouver, Canada.

[6] T. Kohda and A. Tsuneda, "Statistics of chaotic binary sequences," *IEEE Trans. Inform Theory*, vol 43, no. 1, pp. 104-112, 1997.

[7] T. Kohda, "Informations sources using chaotic dynamics," *Proc. of the IEEE*, 2002, vol. 90, pp. 641-661.