2004 International Symposium on Nonlinear
Theory and its Applications (NOLTA2004)
Fukuoka, Japan, Nov. 29 - Dec. 3, 2004

# Design of Nonlinear Control Using Formal Linearization of Polynomial Type

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**Abstract**—In this paper we consider a novel approach of a control design for single-input nonlinear systems. This approach employs a formal linearization method of polynomial type so as to apply linear control theories. Nonlinear systems are linearized with respect to a linearization function which consists of polynomials, and then utilized the linear feedback control law. As a result, the designing of this nonlinear control is reduced to a problem of pole placement in the linear control theories. Numerical experiments indicate that the approach is effective in the face of significant nonlinearities. Modeling errors caused by the linearization can be reduced as the order of linearization function increases.

## 1. Introduction

Studies of nonlinear control systems have been energetically done for many years [1]~[10]. To apply the linear system theories, a linearization based on Taylor expansion truncating up to the first order is well known as easy way to design nonlinear control. But this linearization is not effective for wider region of state space because of poor approximation. It has been reported [8, 9, 10] that a formal linearization with higher order polynomials has excellent accuracy in approximation and is easy in execution by computers.

In this paper, we consider a novel approach for a design of nonlinear control for single-input nonlinear systems using a formal linearization method of polynomial type. Introducing a linearization function which consists of polynomials, the input function is assumed to be linear with respect to the linearization function. We derive the derivative of the linearization function along the trajectories of single-input nonlinear systems and exploit Taylor expansion to them. By truncation of the polynomials, a formal linear systems with respect to the linearization function are obtained. To this formal linear system, we apply the linear state feedback control theory [11]. A feedback gain is decided by pole placement so as to achieve the desired pole locations of linearized systems. As a result, a problem of designing nonlinear control is reduced to a problem of pole placement in the linear control theory by making use of the formal linearization method.

Numerical experiments are included to demonstrate the method and show that the approach is effective in the face of significant nonlinearities. By simulations, model errors caused by the linearization can be reduced by increasing the order of linearization function.

## 2. Control design via formal linearization

Assume that a single-input nonlinear system is given by

$$\Sigma_1 \quad : \quad \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{b}u, \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0 \in D \subset R^n \tag{1}$$

where $\cdot = d/dt$, $\boldsymbol{x} = [x_1, x_2, \cdots, x_n]^T$ is a state vector, $\boldsymbol{f} \in C^N$ is a nonlinear function, $\boldsymbol{b}$ is a constant vector and $u$ is a single input.

In this paper we exploit a formal linearization of polynomial type [8, 9, 10] using Taylor expansion truncating up to the $N$-th order. We define an $N$-th order formal linearization function by

$$\begin{aligned}\boldsymbol{\phi}(\boldsymbol{x}) = [&T_{(10\cdots0)}(\boldsymbol{x}), T_{(01\cdots0)}(\boldsymbol{x}), \cdots, T_{(0\cdots01)}(\boldsymbol{x}),\\ &T_{(11\cdots0)}(\boldsymbol{x}), T_{(101\cdots0)}(\boldsymbol{x}), \cdots, T_{(10\cdots1)}(\boldsymbol{x}),\\ &T_{(20\cdots0)}(\boldsymbol{x}), T_{(21\cdots0)}(\boldsymbol{x}), \cdots, T_{(r_1\cdots r_n)}(\boldsymbol{x}),\\ &\cdots, T_{(N\cdots N)}(\boldsymbol{x})]^T\end{aligned} \tag{2}$$

where

$$T_{(r_1\cdots r_n)}(\boldsymbol{x}) = \prod_{i=1}^{n} x_i^{r_i}.$$

For this linearization function, we suppose that the control is well given by a closed-loop form :

$$u = \boldsymbol{k}\boldsymbol{\phi}(\boldsymbol{x}) \tag{3}$$

where $\boldsymbol{k}$ is a feedback gain :

$$\boldsymbol{k} = [k_1, \cdots, k_{(N+1)^n-1}].$$

We linearize the given nonlinear system (Eq. (1)) into linear system with respect to the linearization function, and the gain $\boldsymbol{k}$ is decided by pole placement using the linear feedback control law so as to achieve the desired pole locations of the linearized system [11].

The derivative of the element of the $\boldsymbol{\phi}$ is

$$\dot{\phi}_\alpha(\boldsymbol{x}) = \dot{T}_{(r_1\cdots r_n)}(\boldsymbol{x})$$

$$= (\frac{d}{dt}x_1^{r_1})x_2^{r_2}\cdots x_{n-1}^{r_n-1}x_n^{r_n} + \cdots + x_1^{r_1}x_2^{r_2}\cdots x_{n-1}^{r_n-1}(\frac{d}{dt}x_n^{r_n})$$

$$= \sum_{\ell=1}^{n} r_\ell(f_\ell(\boldsymbol{x}) + b_\ell\boldsymbol{k}\boldsymbol{\phi}(\boldsymbol{x}))\frac{T_{(r_1\cdots r_n)}(\boldsymbol{x})}{x_\ell}, \quad (4)$$

$$\alpha = \alpha(r_1, \cdots, r_n).$$

Note that Taylor expansion up to the $N$-th order derives

$$f_\ell(\boldsymbol{x}) + b_\ell\boldsymbol{k}\boldsymbol{\phi}(\boldsymbol{x}) = [g_{\ell1}, g_{\ell2}, \cdots, g_{\ell(N+1)^n-1}]\boldsymbol{\phi}(\boldsymbol{x}) + b_\ell\boldsymbol{k}\boldsymbol{\phi}(\boldsymbol{x}) + \text{higher order} \quad (5)$$

where

$$g_{\ell j} = \frac{\partial^{(r_1+r_2+\cdots+r_n)}}{\partial x_1^{r_1}\partial x_2^{r_2}\cdots\partial x_n^{r_n}} f_\ell(\boldsymbol{x})\,|_{\boldsymbol{x}=0},$$

$$j = j(r_1, \cdots, r_n).$$

From Eqs.(4) and (5), it follows that

$$\begin{aligned}
\dot{\phi}_\alpha(\boldsymbol{x}) &= r_1[g_{11} + b_1k_1,\ g_{12} + b_1k_2, \cdots,\\
&g_{1(N+1)^n-1} + b_1k_{(N+1)^n-1}]\boldsymbol{\phi}(\boldsymbol{x})x_1^{r_1-1}x_2^{r_2}\cdots x_n^{r_n} +\\
&r_2[g_{21} + b_2k_1, g_{22} + b_2k_2, \cdots,\\
&g_{2(N+1)^n-1} + b_2k_{(N+1)^n-1}]\boldsymbol{\phi}(\boldsymbol{x})x_1^{r_1}x_2^{r_2-1}\cdots x_n^{r_n} +\\
&\vdots\\
&r_n[g_{n1} + b_nk_1, g_{n2} + b_nk_2, \cdots,\\
&g_{n(N+1)^n-1} + b_nk_{(N+1)^n-1}]\boldsymbol{\phi}(\boldsymbol{x})x_1^{r_1}x_2^{r_2}\cdots x_n^{r_n-1}\\
&+ \text{higher order}\\
&= [G_{\alpha1}(\boldsymbol{k}), \cdots, G_{\alpha\beta}(\boldsymbol{k}), \cdots, G_{\alpha(N+1)^n-1}(\boldsymbol{k})]\boldsymbol{\phi}(\boldsymbol{x})\\
&+\epsilon_\alpha(\boldsymbol{k}, \boldsymbol{x}) \quad (6)
\end{aligned}$$

where

$$G_{\alpha\beta}(\boldsymbol{k}) = \sum_{\ell=1}^{n} r_\ell\{g_{\ell\,\beta(q_1-r_1,\cdots,q_\ell-r_\ell+1,\cdots,q_n-r_n)} + b_\ell k_{\beta(q_1-r_1,\cdots,q_\ell-r_\ell+1,\cdots,q_n-r_n)}\}, \quad (7)$$

$$g_{\ell\beta} + b_\ell k_\beta = \begin{cases} g_{\ell j} + b_\ell k_j & (\beta = j) \\ 0 & (\beta \neq j) \end{cases}.$$

The $\epsilon_\alpha(\boldsymbol{k}, \boldsymbol{x})$ is the error term whose order includes higher than $N$ with respect to some $x_i$ $(i = 1, \cdots, n)$. By considering $\epsilon_\alpha(\boldsymbol{k}, \boldsymbol{x})$ being model error noise, a formal linear system is derived by

$$\Sigma_2 : \dot{\boldsymbol{\phi}}(\boldsymbol{x}) = A(\boldsymbol{k})\boldsymbol{\phi}(\boldsymbol{x}) + \boldsymbol{\epsilon}(\boldsymbol{k}, \boldsymbol{x}), \quad (8)$$

$$A(\boldsymbol{k}) = [G_{ij}(\boldsymbol{k})] \in R^{\{(N+1)^n-1\}\times\{(N+1)^n-1\}},$$

$$\boldsymbol{\epsilon}(\boldsymbol{k}, \boldsymbol{x}) = [\epsilon_j(\boldsymbol{k}, \boldsymbol{x})] \in R^{(N+1)^n-1}.$$

We would like to find such feedback gain $\boldsymbol{k}$ as the linearized system (Eq. (8)) approaches to zero as small as possible when $t \to \infty$. Computer procedure is as follows.

At first, we initialize the $\boldsymbol{k}$ as $\boldsymbol{k} = 0$. Since $\boldsymbol{\epsilon}(\boldsymbol{k}, \boldsymbol{x})$ is error function caused by approximation, $k_i^*$ $(i = n+1, \cdots, (N+1)^n - 1)$ are found so as to minimize the norm of $\boldsymbol{\epsilon}(\boldsymbol{k}, \boldsymbol{x})$ :

$$\{k_{n+1}^*, \cdots, k_{(N+1)^n-1}^*\} = \{\min_{\{k_{n+1},\cdots,k_{(N+1)^n-1}\}} \sup_{\{|x_i|,(i=1,\cdots,n)\}} \|\boldsymbol{\epsilon}(\boldsymbol{k}, \boldsymbol{x})\|\,|_{\{k_i=0,\ (i=1,\cdots,n)\}}\} \quad (9)$$

where $\|\cdot\|$ is Euclidean norm. Next, $k_i^*$ $(i = 1, \cdots, n)$ is chosen so that all poles of the matrix $A(\boldsymbol{k})$ have negative real parts on the s-plane by pole placement to achieve the desired pole locations $s_i$ $(i = 1, \cdots, n)$ :

$$\{k_1^*, \cdots, k_n^*\} = \{\text{Poles of } A(k_1, k_2, \cdots, k_n, k_{n+1}^*, \cdots, k_{(N+1)^n-1}^*) = s_i, (i = 1, \cdots, n)\} \quad (10)$$

where

$$\text{Real part of } s_i < 0.$$

As a result, we can construct a nonlinear control as following steps.

**<Algorithm>**

**step 1** Apply a formal linearization to the given nonlinear system and obtain A($\boldsymbol{k}$) and $\boldsymbol{\epsilon}(\boldsymbol{k},\boldsymbol{x})$ in Eq. (8) .

**setp 2** Set $\boldsymbol{k} = 0$.

**step 3** Find $\{k_{n+1}^*, \cdots, k_{(N+1)^n-1}^*\}$ by minimizing the norm of $\boldsymbol{\epsilon}(\boldsymbol{k}, \boldsymbol{x})$ in Eq. (9) and fix them.

**step 4** Decide poles $s_i$ $(i = 1, \cdots, n)$ and find $\{k_1^*, \cdots, k_n^*\}$ in Eq. (10), and then fix them.

**step 5** Construct a closed-loop control $u = \boldsymbol{k}^*\boldsymbol{\phi}(\boldsymbol{x})$ .

This method approximates the derivative of the linearization function based on Taylor expansion truncating up to the $N$-th order of each state variable $x_i$ $(i = 1, \cdots, n)$. Let $\epsilon_F(\boldsymbol{x})$ be the norm of the error function in Eq. (8) when $\boldsymbol{k} = \boldsymbol{k}^*$ :

$$\begin{aligned}
\epsilon_F(\boldsymbol{x}) &= \|\,(\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{b}\boldsymbol{k}^*\boldsymbol{\phi}(\boldsymbol{x})) - A(\boldsymbol{k}^*)\boldsymbol{\phi}(\boldsymbol{x})\,\|\\
&= \|\,\boldsymbol{\epsilon}(\boldsymbol{k}^*, \boldsymbol{x})\,\| . \quad (11)
\end{aligned}$$

In case of a linearization based on Taylor expansion up to the first order, a model error norm $\epsilon_T(\boldsymbol{x})$ is defined by

$$\begin{aligned}
\epsilon_T(\boldsymbol{x}) &= \|\,(\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{b}u) - \{(\frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}^T})\,|_{\boldsymbol{x}=\boldsymbol{x}_s}\,(\boldsymbol{x} - \boldsymbol{x}_s) + \boldsymbol{b}u\}\,\|\\
&= \|\,\boldsymbol{f}(\boldsymbol{x}) - (\frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}^T})\,|_{\boldsymbol{x}=\boldsymbol{x}_s}\,(\boldsymbol{x} - \boldsymbol{x}_s)\,\| \quad (12)
\end{aligned}$$

where $\boldsymbol{x}_s$ is an operating point.

## 3. Numerical examples

We illustrate numerical experiments of the above proposed nonlinear control and the conventional control via linearization of the first order Taylor expansion method for comparison.

### 3.1. Example 1

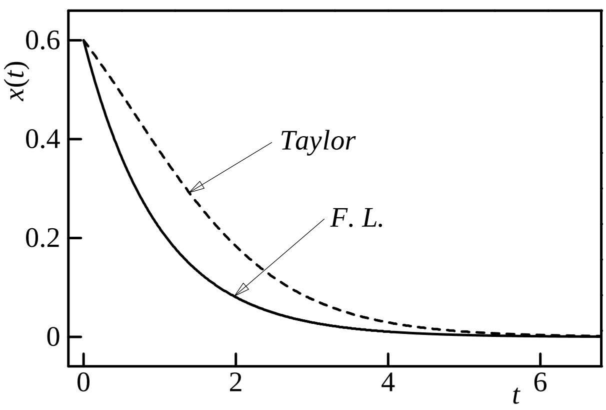


Figure 1: Responses for the scalar system

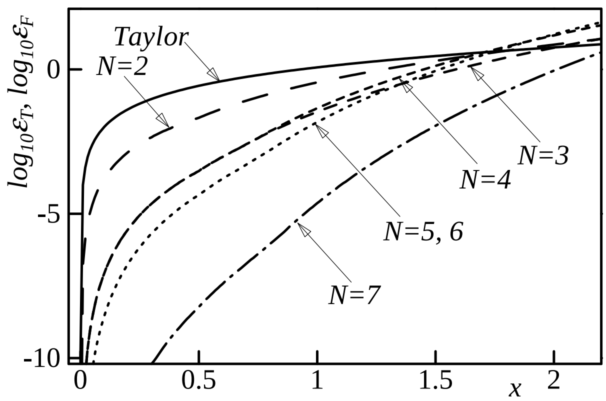


Figure 2: Model errors for the scalar system

We consider the following scalar system :

$$\dot{x} = e^{-x} - \cos(x) + u \ , \quad D = [0, 50]. \qquad (13)$$

This nonlinear system (Eq. (13)) is transformed into the formal linear system (Eq. (8)). By minimizing the norm of $\boldsymbol{\epsilon}(\boldsymbol{k}, x)$ in Eq. (9) ,

$$\{k_2^* = -1,\ k_3^* = -2,\ k_4^* = 0,\ k_5^* = \frac{-1}{5},\ k_6^* = \frac{-2}{6},\ k_7^* = \frac{-1}{7}\}$$

are found when $N = 7$. We decide a pole of $A(\boldsymbol{k})$ in Eq. (8) as

$$s_1 = -1$$

so that

$$k_1^* = -2$$

is set. Using this feedback gain $\boldsymbol{k}^*$, we construct a nonlinear control $u$ (Eq. (3)). Fig. 1 shows time responses $x(t)$ using this $u$ by the formal linearization (*F. L.*) when $N = 4$, and by the conventional first order Taylor expansion (*Taylor*) when a pole is set the same value $\{-1\}$ and $x_s = 0$.

Fig. 2 shows the model errors $\epsilon_F(x)$ of Eq. (11) when the order of linearization function is varied from $N = 2$ to 7, and $\epsilon_T(x)$ of Eq. (12).

Table 1 shows the maximum points of stable region in which $x(t) \to 0$ as $t \to \infty$.

Table 1: Stable regions for the scalar system

| *Taylor* | *N*=2 | *N*=3 | *N*=4 | *N*=5 | *N*=6 | *N*=7 |
|---|---|---|---|---|---|---|
| 0.86 | 2.07 | 2.74 | 2.94 | 3.09 | 3.90 | 4.50 |

### 3.2. Example 2

We consider the following two-dimensional system :

$$\begin{cases} \dot{x}_1(t) = x_2 \\ \dot{x}_2(t) = -x_2 + x_1^2 x_2^2 + 2x_1^3 + x_2^4 + u \end{cases} , \qquad (14)$$

$$D = [-100, 100] \times [-100, 100].$$

This nonlinear system (Eq. (14)) is transformed into the formal linear system (Eq. (8)). By minimizing the norm of $\boldsymbol{\epsilon}(\boldsymbol{k}, \boldsymbol{x})$ in Eq. (9),

$$k_i^* = \begin{cases} -2 & (i = \alpha(2,2), \alpha(3,0)) \\ 0 & (otherwise) \end{cases}$$

are found when $N = 3$. We decide poles of $A(\boldsymbol{k})$ in Eq. (8) as

$$s_1 = s_2 = -1$$

so that

$$k_1^* = -1 \ \ and \ \ k_2^* = -1$$

are set. Using this feedback gain $\boldsymbol{k}^*$, we construct a nonlinear control $u$ (Eq. (3)). Figs. 3 and 4 show time responses of $x_1$ and $x_2$, respectively, which compare between the formal linearization (*by F. L.*) and Taylor expansion (*by Taylor*) when poles are set the same values $\{-1, -1\}$ and $\boldsymbol{x}_s = 0$. Fig. 5 depicts the stable regions in these cases.

## 4. Conclusions

We have considered a new approach for a design of nonlinear control for single-input nonlinear systems using a formal linearization method to apply linear control theories. By this approach, a problem of designing nonlinear control is able to be reduced to a problem of pole placement applying the linear control theories. The numerical examples confirm that this approach is better in accuracy than the conventional one like the linearization based on Taylor expansion. These results have illustrated that the model error caused by approximation using this method could be improved as the order of formal linearization function increases. It is left in future works to solve problems of cases such as 1) multi-input system, 2) more general nonlinear system, and 3) another formal linearization like Chebyshev polynomials.

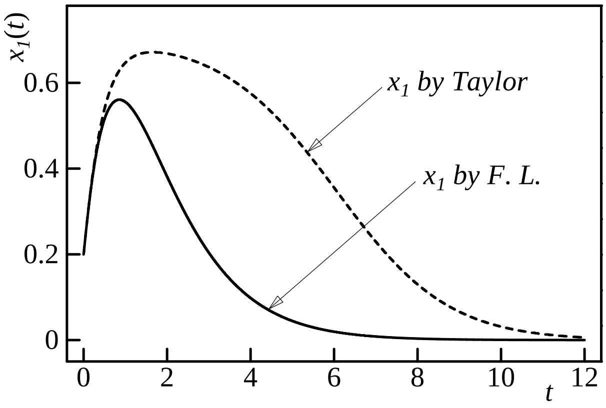


Figure 3: Responses $x_1(t)$ for the two-dimensional system

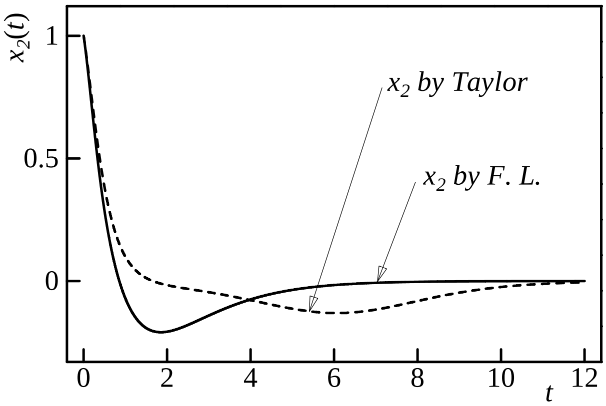


Figure 4: Responses $x_2(t)$ for the two-dimensional system

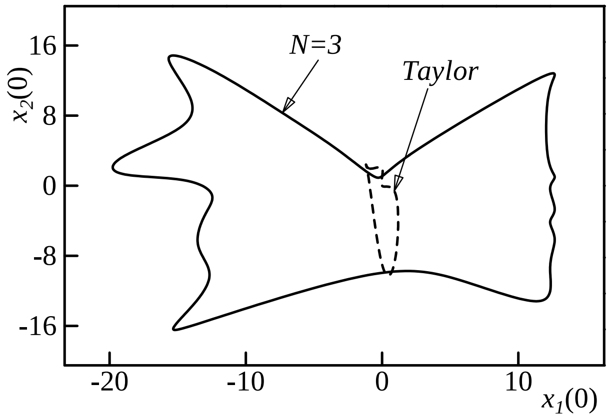


Figure 5: Stable regions for the two-dimensional system

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