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Amplitude Death Induced by a Global Dynamic Coupling

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Abstract—Amplitude death is well known as a coupling induced stabilization of fixed points in coupled oscillators. This paper proposes a global dynamic coupling that can induce amplitude death. Linear stability analysis is used to derive a stability condition for this death. It is shown that the odd number property of the delayed feedback control exists in globally coupled oscillators. Furthermore, van der Pol electrical oscillators coupled by the well-known RC line (T-type) are used to verify the theoretical results.

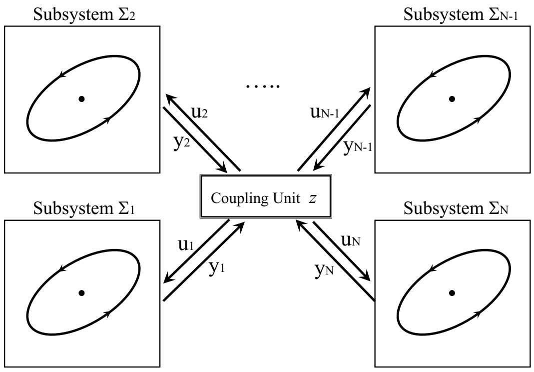

Figure 1: Globally coupled oscillators

1. Introduction

Oscillations can cease in diffusive coupled oscillators that have large variances in their distributed frequencies. This phenomenon is known as *amplitude death* or *oscillation death*, and has been studied in the field of nonlinear physics [1]. Reddy *et al.* [2, 3] found that two identical oscillators coupled with a *time delay* can cause amplitude death. Time delay induced death has received considerable attention [4]. A sufficient condition under which death does not occur was derived [5, 6].

It was recently discovered that amplitude death in two coupled identical oscillators can be induced by incorporating a dynamic coupling without a time delay [7, 8]. From these papers, the following results were obtained. First, amplitude death was observed in both the numerical simulations and experiments. In addition, a sufficient condition for avoiding death in m-dimensional oscillators was found and a necessary and sufficient condition for death in two-dimensional simple oscillators was derived.

This paper proposes a *globally* dynamic coupling that can induce amplitude death. This death, however, does not occur in the globally coupled oscillators if the Jacobi matrix evaluated at the fixed point satisfies an odd-number property. The coupling presented here differs slightly from that in a previous paper [7], although both types of coupling have features in common. They each have their own dynamics, and the coupling signals converge to zero when amplitude death occurs. Experiments verify amplitude death in van der Pol oscillators coupled by a global dynamic connection.

2. Coupled Oscillators

Consider N identical m-dimensional oscillators

$$\Sigma_j : \begin{cases} \dot{\boldsymbol{x}}_j &= \boldsymbol{F}(\boldsymbol{x}_j) + \boldsymbol{b}u_j \\ y_j &= \boldsymbol{c}\boldsymbol{x}_j, \end{cases} \quad (j = 1, 2, \ldots, N)$$

where $\boldsymbol{x}_j \in \boldsymbol{R}^m$ is the system variable, $u_j \in \boldsymbol{R}$ and $y_j \in \boldsymbol{R}$ are the input and output signals, respectively, as shown in Fig. 1. $\boldsymbol{F} : \boldsymbol{R}^m \to \boldsymbol{R}^m$ is a continuously differentiable nonlinear function. The input and output vectors are denoted by $\boldsymbol{b} \in \boldsymbol{R}^m$ and $\boldsymbol{c} \in \boldsymbol{R}^{1\times m}$. Each individual oscillator is assumed to have an unstable fixed point $\boldsymbol{x}_f$ (i.e., $\boldsymbol{F}(\boldsymbol{x}_f) = \boldsymbol{0}$) that is surrounded by oscillatory behavior.

Two types of global diffusive coupling are proposed: static and dynamic. Static coupling is denoted as

$$u_j = k\left(\frac{1}{N}\sum_{l=1}^{N} y_l - y_j\right), \tag{1}$$

where $k \in \boldsymbol{R}$ is the coupling strength. The input signal u_j is proportional to the difference between the average value of all the output signals and its own signal y_j. The steady state of the coupled oscillators is described by

$$\begin{bmatrix}\boldsymbol{x}_1^T & \boldsymbol{x}_2^T & \cdots & \boldsymbol{x}_N^T\end{bmatrix}^T = \begin{bmatrix}\boldsymbol{x}_f^T & \boldsymbol{x}_f^T & \cdots & \boldsymbol{x}_f^T\end{bmatrix}^T. \tag{2}$$

On the other hand, the oscillators Σ_j ($j = 1, 2, \ldots, N$) for the dynamic coupling are coupled by

$$\dot{z} = \gamma\left(\sum_{l=1}^{N} y_l - Nz\right), \quad u_j = k(z - y_j). \tag{3}$$

Here, the input signal u_j is proportional to the difference between $z \in \boldsymbol{R}$ and y_j, where z is an additional variable governed by dynamical equation (3). $\gamma > 0$ is a parameter. It should be noted that this coupling differs from that described in a previous paper [7]. The steady state of the dynamic coupled system is described by

$$\begin{bmatrix}\boldsymbol{x}_1^T & \cdots & \boldsymbol{x}_N^T & z\end{bmatrix}^T = \begin{bmatrix}\boldsymbol{x}_f^T & \cdots & \boldsymbol{x}_f^T & \boldsymbol{c}\boldsymbol{x}_f\end{bmatrix}^T. \quad (4)$$

For both types of coupling, u_j $(j = 1, 2, \ldots, N)$ become zero if the system variables $\boldsymbol{x}_j$ $(j = 1, 2, \ldots, N)$ are synchronized. Neither type changes the location of fixed point $\boldsymbol{x}_f$, since static coupling (1) and dynamic coupling (3) are both diffusive [1]. Amplitude death can be defined as a diffusive-coupling induced steady state stabilization.

3. Linear Stability Analysis

Let $\boldsymbol{x}_j = \boldsymbol{x}_f + \boldsymbol{X}_j$ $(j = 1, 2, \ldots, N)$, where $\boldsymbol{X}_j$ is assumed to be small. The linearized subsystems,

$$\Delta\Sigma_j : \left\{ \begin{array}{lcl} \dot{\boldsymbol{X}}_j & = & \boldsymbol{A}\boldsymbol{X}_j + \boldsymbol{b}U_j \\ Y_j & = & \boldsymbol{c}\boldsymbol{X}_j \end{array} \right. \quad (j = 1, 2, \ldots, N),$$

are obtained by substituting $\boldsymbol{x}_j$ into oscillators Σ_j. In these equations, $Y_j = y_j - \boldsymbol{c}\boldsymbol{x}_f$. The Jacobi matrix of the nonlinear function $\boldsymbol{F}$ is given by $\boldsymbol{A} := \{\partial \boldsymbol{F}(\boldsymbol{x})/\partial \boldsymbol{x}\}_{\boldsymbol{X}=\boldsymbol{x}_f}$. We assume that $\boldsymbol{A}$ does not have an eigenvaule on the orgin. The linearized subsystems $\Delta\Sigma_j$ are then coupled by

$$U_j = k\left(\frac{1}{N}\sum_{l=1}^{N} Y_l - Y_j\right), \quad (5)$$

for the static coupling. On the other hand, they are coupled with

$$\dot{Z} = \gamma\left(\sum_{l=1}^{N} Y_l - NZ\right), \quad U_j = k(Z - Y_j), \quad (6)$$

for the dynamic coupling, where $Z = z - \boldsymbol{c}\boldsymbol{x}_f$.

3.1. Static Coupling

The linear stability of steady state (2) in the static coupled system is equivalent to that in the linearized subsystems $\Delta\Sigma_j$ with connection (5). As a result, the closed loop system consisting of $\Delta\Sigma_j$ and (5),

$$\dot{\boldsymbol{X}}_j = \boldsymbol{A}\boldsymbol{X}_j + \boldsymbol{b}k\boldsymbol{c}\left\{\frac{1}{N}\sum_{l=1}^{N} \boldsymbol{X}_l - \boldsymbol{X}_j\right\} \quad (j = 1, 2, \ldots, N), \quad (7)$$

is considered. From linear system (7), the following Lemma is derived.

Lemma 1. *Suppose $\boldsymbol{A}$ is an unstable matrix. Static coupling (1) never induces amplitude death in the coupled oscillators.*

Proof. System (7) can be rewritten as

$$\begin{bmatrix}\dot{\boldsymbol{X}}_1 \\ \dot{\boldsymbol{X}}_2 \\ \vdots \\ \dot{\boldsymbol{X}}_N\end{bmatrix} = \begin{bmatrix}\overline{\boldsymbol{A}} & \overline{\boldsymbol{b}} & \cdots & \overline{\boldsymbol{b}} \\ \overline{\boldsymbol{b}} & \overline{\boldsymbol{A}} & \cdots & \overline{\boldsymbol{b}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{\boldsymbol{b}} & \overline{\boldsymbol{b}} & \cdots & \overline{\boldsymbol{A}}\end{bmatrix}\begin{bmatrix}\boldsymbol{X}_1 \\ \boldsymbol{X}_2 \\ \vdots \\ \boldsymbol{X}_N\end{bmatrix}. \quad (8)$$

where $\overline{\boldsymbol{A}} := \boldsymbol{A} - \boldsymbol{b}k\boldsymbol{c}$ and $\overline{\boldsymbol{b}} := \frac{1}{N}\boldsymbol{b}k\boldsymbol{c}$. The characteristic function of linear system (8) is given by $f(\lambda) = Nf_1(\lambda)f_2(\lambda)$, where

$$f_1(\lambda) := \det[\lambda \boldsymbol{I}_m - \boldsymbol{A}], \quad f_2(\lambda) := \det\left[\lambda \boldsymbol{I}_m - \boldsymbol{A} + \left(1 + \frac{1}{N}\right)\boldsymbol{b}k\boldsymbol{c}\right].$$

It is obvious that $f_1(\lambda)$ and $f_2(\lambda)$ are the characteristic functions of (8). Since $\boldsymbol{A}$ is unstable, $f_1(\lambda) = 0$ has at least one root in the open right-half of the complex plane. This fact guarantees that static coupling (1) never induces steady state stabilization, that is, amplitude death. □

3.2. Dynamic Coupling

The linear stability of steady state (4) in the dynamic coupled systems is equivalent to that in the linearized subsystems $\Delta\Sigma_j$ with connection (6). The closed loop system consisting of $\Delta\Sigma_j$ and (6) is

$$\dot{\boldsymbol{X}}_j = \boldsymbol{A}\boldsymbol{X}_j + \boldsymbol{b}k(Z - \boldsymbol{c}\boldsymbol{X}_j) \quad (j = 1, 2, \ldots, N), \quad (9a)$$

$$\dot{Z} = \gamma\left(\boldsymbol{c}\sum_{l=1}^{N} \boldsymbol{X}_l - NZ\right). \quad (9b)$$

The necessary and sufficient condition for system (9) to be stable is provided in the following lemma.

Lemma 2. *Steady state (4) for the dynamic coupled oscillators is stable if and only if both*

$$\boldsymbol{A} - \boldsymbol{b}k\boldsymbol{c}, \quad \begin{bmatrix}\boldsymbol{A} - \boldsymbol{b}k\boldsymbol{c} & N\boldsymbol{b}k \\ \gamma\boldsymbol{c} & -\gamma N\end{bmatrix} \quad (10)$$

are stable matrices.

Proof. System (9) can be rewritten as

$$\begin{bmatrix}\dot{\boldsymbol{X}}_1 \\ \dot{\boldsymbol{X}}_2 \\ \vdots \\ \dot{\boldsymbol{X}}_N \\ \dot{Z}\end{bmatrix} = \begin{bmatrix}\overline{\boldsymbol{A}} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{b}k \\ \boldsymbol{0} & \overline{\boldsymbol{A}} & \cdots & \boldsymbol{0} & \boldsymbol{b}k \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \overline{\boldsymbol{A}} & \boldsymbol{b}k \\ \gamma\boldsymbol{c} & \gamma\boldsymbol{c} & \cdots & \gamma\boldsymbol{c} & -\gamma N\end{bmatrix}\begin{bmatrix}\boldsymbol{X}_1 \\ \boldsymbol{X}_2 \\ \vdots \\ \boldsymbol{X}_N \\ Z\end{bmatrix}. \quad (11)$$

The characteristic function of linear system (11) is given by $g(\lambda) = Ng_1(\lambda)g_2(\lambda)$, where

$$g_1(\lambda) := \det\left[\lambda \boldsymbol{I}_m - \overline{\boldsymbol{A}}\right], \quad g_2(\lambda) := \det\begin{bmatrix}\lambda \boldsymbol{I}_m - \overline{\boldsymbol{A}} & -N\boldsymbol{b}k \\ -\gamma\boldsymbol{c} & \lambda + \gamma N\end{bmatrix}$$

$g_1(\lambda)$ and $g_2(\lambda)$ are the characteristic functions of the matrices in (10). □

Theorem 1. *Steady state (4) for the dynamic coupled oscillators is unstable, that is, amplitude death never occurs for any $\boldsymbol{b}, k, \boldsymbol{c}$, if $\boldsymbol{A}$ has an odd-number of real positive eigenvalues (odd-number property).*

Proof. If the two conditions, $\lim_{\lambda\to\infty} g_2(\lambda) = \infty$ and $g_2(0) < 0$, are satisfied, at least one root of $g_2(\lambda) = 0$ is in the open right-half of the complex plane (i.e., steady state (4) is unstable). The first condition always holds, and the second condition can be described by

$$g_2(0) = N\gamma \det\left[-\boldsymbol{A}\right] = N\gamma \prod_{q=1}^{m} (-\sigma_q),$$

where σ_q $(q = 1, 2, \ldots, m)$ are the eigenvalues of $\boldsymbol{A}$. Hence, if $\boldsymbol{A}$ has an odd-number of real positive eigenvalues (*odd-number property*), then $g_2(0) < 0$ is satisfied. □

The odd-number property is well known in the field of delayed feedback control of chaos. A similar stability analysis can be found in [9, 10].

4. van der Pol Oscillators

Coupled van der Pol oscillators have been used as a typical coupled system for analytical and experimental studies in direct coupling [11], capacitance coupling [12], and inductance coupling [13].

4.1. Stability Analysis

Consider N identical van der Pol oscillators, as shown in Fig. 2. With v_j representing the voltage across the capacitance C and i_j representing the current through inductor L of the i-th oscillator, the model for the van der Pol oscillators is written as

$$\begin{cases} L\dfrac{\mathrm{d}i_j}{\mathrm{d}t} = v_j \\ C\dfrac{\mathrm{d}v_j}{\mathrm{d}t} = -i_j - h(v_j) + \dfrac{1}{R}(v_0 - v_j) \end{cases} \quad (j = 1, 2, \ldots, N), \tag{12}$$

where current $h(v_j)$ goes through the nonlinear resistor

$$h(v) = \begin{cases} \mu_2 v - B_p(\mu_1 + \mu_2) & v \geq +B_p \\ -\mu_1 v & |v| \leq B_p \\ \mu_2 v + B_p(\mu_1 + \mu_2) & v \leq -B_p \end{cases},$$

with $\mu_1 > 0$ and $\mu_2 > 0$. When switch S is open in the coupling unit, the variable v_0 does not have its own dynamics: $v_0 = \left(\sum_{j=1}^{N} v_j\right)/N$. Hence, v_0 is the average voltage of v_j $(j = 1, 2, \ldots, N)$. This corresponds to the static coupled system consisting of oscillators Σ_j and coupling (1). Using Lemma 1, the following result can be obtained easily.

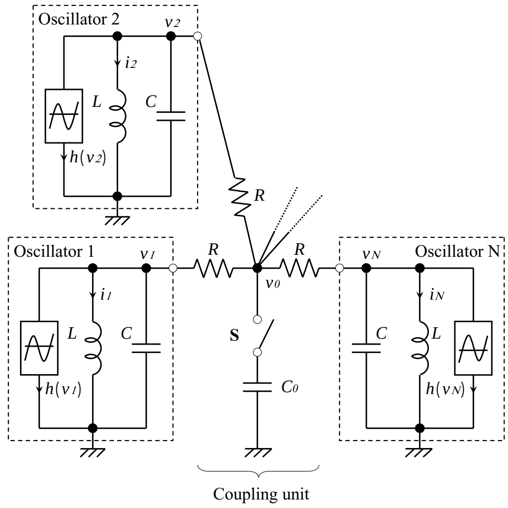

Figure 2: Globally coupled van der Pol oscillators

Corollary 1. *If oscillators (12) are coupled only by a resistor (S is open), then amplitude death never occurs (i.e., steady state (4) is unstable) for any $R > 0$.*

On the other hand, when switch S is closed, variable v_0 does have its own dynamics. These dynamics are given by

$$C_0 \frac{\mathrm{d}v_0}{\mathrm{d}t} = \frac{1}{R}\left(\sum_{l=1}^{N} v_l - Nv_0\right), \tag{13}$$

where C_0 is the coupling capacitance. The oscillators with an RC line (T-type) corresponds to the dynamic coupled system consisting of oscillators Σ_j and coupling (3). This RC coupling is a model for describing the mutual coupling effect of electrical wires in large scale integrated circuits [14].

The Routh-Hurwitz criterion and Lemma 2 provide us a necessary and sufficient condition for the stability of steady state (4).

Corollary 2. *Consider oscillators (12) with RC coupling (13). Steady state (4) is stable if and only if the circuit parameters satisfy*

$$\begin{aligned} &0 < \frac{1}{R} - \mu_1, \; 0 < C_0 R - NL\mu_1, \\ &0 < C_0(1 - \mu_1 R)(RC_0 - NL\mu_1) - N^2 LC\mu_1. \end{aligned} \tag{14}$$

4.2. Implementation

The coupled circuits shown in Fig. 2 were constructed and their performance was evaluated. The circuit parameters were set to $L = 22$ [mH], $C = 0.1$ [μF], $\mu_1 = 1.0\times10^{-3}$,

$\mu_2 = 1.0 \times 10^{-3}$, and $B_p = 3.0$[V]. The nonlinear resistor has the same structure as in [12] and contains operational amplifier (TL084) and three resistors (1.0 [kΩ]). Figure 3 indicates implemented coupled circuits.

The results for the $R - C_0$ parameter region, in which steady state (4) is stable based on stability condition (14), were estimated. Figures 4 (a) and (b) show the region for $N = 2$ and $N = 8$, respectively. The dots indicate the parameter where death is observed in the coupled electronic circuits. The solid line presents the theoretical amplitude death boundary. It is evident that the region shrinks as the number of subsystems N increases. These experimental results agree well with the stability analysis described in the previous section.

5. Conclusion

In this study, global dynamic coupled van der Pol oscillators were implemented and the experimental amplitude death observations were presented. The circuit elements used in the experiments are relatively inexpensive and can be easily found. Therefore, it has been demonstrated experimentally that death induced by globally dynamic coupling is a robust phenomenon for external noise and mismatched parameters.

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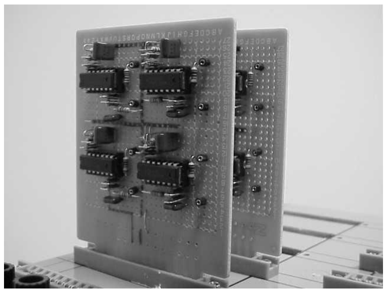
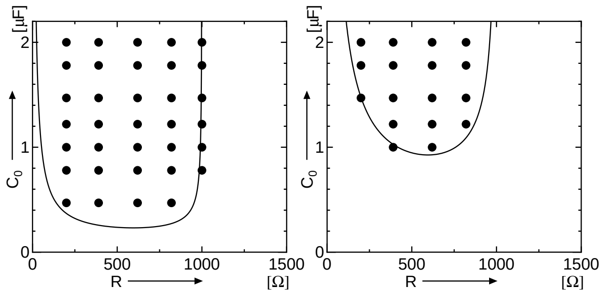


Figure 3: Implementation of eight van der Pol circuits



(a) N=2 (a) N=8

Figure 4: Parameter region of stable steady state for $N = 2$ and 8. The solid line indicates the theoretical amplitude death boundary. The dot represents the parameters where death is observed in the real circuits.

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