2004 International Symposium on Nonlinear
Theory and its Applications (NOLTA2004)
Fukuoka, Japan, Nov. 29 - Dec. 3, 2004

Detection of Periodic Oscillations Using Homotopy with Complex State Variables

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Abstract—This paper describes a method to detect periodic oscillations in a circuit parameter space. It is a difficult problem to seek the region of the target oscillation in a parameter space, especially if the region is very small. To overcome the difficulty, we obtain complex periodic solutions by extending the state variables of the circuit equation in real number field. The extension makes it possible to find the solutions easily. After finding the complex solutions numerically, we detect the real solutions, by a general homotopy. In this stage, we use the fact that the complex general homotopy is monotonic in the parameter space.

1. Introduction

When we analyze nonlinear circuits, it is important to find circuit parameters on which the target oscillations are generated. Once a solution of the determining equation of the target oscillations is detected, we can apply general homotopy to trace the solutions. Many researchers have been studied about finding all solutions on a fixed circuit parameters, e.g., homotopy method[1], interval method[2, 3, 4], and so on. However, in order to detect the any solutions, we have to solve the equation a number of times in the parameter space through a trial and error process. To overcome the difficulty of the detection, we propose to extend the real state variables to complex numbers on circuit equations. Although it is difficult to seek the region of real solutions, it is easier to find complex solutions on a fixed parameter since there exist complex solutions in the region where the real solution does not exist.

Complex function which is continuously differentiable has many fascinating properties. Based on the Cauchy-Riemann equations, the homotopy path of the complex function becomes monotonic in homotopy parameters[1]. Using the property, we propose a method to find the target periodic oscillation in the parameter space. That is, we first find all or almost all complex solutions of a determining equation on a fixed circuit parameter. In this stage, we use Newton homotopy. Then, using the general homotopy from the obtained solutions in the parameter space, we can seek the region of real periodic solutions. The monotoneity of the Newton homotopy and the general homotopy makes it possible to detect real solutions efficiently.

In section 2, we define the extension to complex state variables. In section 3, we show the monotoneity of the homotopy path. In section 4, we describe the algorithm for detecting real periodic solutions. We confirm the efficiency of the proposed method by applying it to a RLC-resonance circuit in section 5.

2. Definition of Complex Determining Equation

We consider a scaled circuit equation

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{e}(t) \tag{1}$$

where $\boldsymbol{x} = (x_1, \ldots, x_n)' \in \boldsymbol{R}^n$ is a vector of state variables and the prime means transpose. We assume that $\boldsymbol{f}(\boldsymbol{x}) : \boldsymbol{R}^n \mapsto \boldsymbol{R}^n$ is represented by polynomials of $x_1, \ldots, x_n$. The vector $\boldsymbol{e}(t) \in \boldsymbol{R}^n$ corresponds to AC sources of period 2π:

$$\boldsymbol{e}(t + 2\pi) = \boldsymbol{e}(t). \tag{2}$$

By extending the real state variables to complex number field, we obtain the following equation:

$$\frac{\mathrm{d}\boldsymbol{z}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{z}) + \boldsymbol{e}(t). \tag{3}$$

where $\boldsymbol{z} \in \boldsymbol{C}^n$ and $\boldsymbol{f}(\boldsymbol{z}) : \boldsymbol{C}^n \mapsto \boldsymbol{C}^n$ is represented by polynomials of $z_1, \ldots, z_n$. The time t, the source $\boldsymbol{e}(t)$ and the coefficients of the polynomials are still in real number field.

We consider a periodic solution of Eq.(3). The integration of Eq.(3) from an initial value $\boldsymbol{z}(0) = \boldsymbol{z}_0$ gives

$$\boldsymbol{z}(t) = \boldsymbol{z}_0 + \int_0^t \boldsymbol{f}(\boldsymbol{z}, s)\mathrm{d}s. \tag{4}$$

A problem of finding a periodic solution of period $T \in \boldsymbol{R}$ is a two-point boundary value problem in which the solution of Eq.(3) in the interval $[0, T]$ must satisfy the boundary condition $\boldsymbol{z}(0) = \boldsymbol{z}(T)$. Assuming that we can integrate Eq.(3) from $t = 0$ to $t = T$, we express the above problem using a mapping $\boldsymbol{T} : \boldsymbol{C}^n \mapsto \boldsymbol{C}^n$,

$$\boldsymbol{z}_0 = \boldsymbol{T}(\boldsymbol{z}_0), \quad \boldsymbol{T}(\boldsymbol{z}_0) \equiv \int_0^T \boldsymbol{f}(\boldsymbol{z}, s)\mathrm{d}s + \boldsymbol{z}_0. \tag{5}$$

To solve the two-point boundary value problem with a shooting method, we define a complex nonlinear equation

$$\boldsymbol{F}(\boldsymbol{z}_0) \equiv \boldsymbol{z}_0 - \boldsymbol{T}(\boldsymbol{z}_0) = \boldsymbol{0}. \qquad (6)$$

The solution of Eq.(6), that is, the fixed pint of the mapping $\boldsymbol{T}$ determines a periodic solution of Eq.(3) [5].

In order to clarify the complex analyticity of the function $\boldsymbol{F}(\boldsymbol{z}_0)$, we consider the Jacobi matrix of the function $\boldsymbol{F}(\boldsymbol{z}_0)$:

$$\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{z}_0} = \boldsymbol{1} - \frac{\partial \boldsymbol{T}}{\partial \boldsymbol{z}_0}. \qquad (7)$$

The Jacobi matrix is calculated by integrating a linear equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \boldsymbol{z}(t)}{\partial \boldsymbol{z}_0} = \frac{\partial \boldsymbol{f}(\boldsymbol{z}(t))}{\partial \boldsymbol{z}} \cdot \frac{\partial \boldsymbol{z}(t)}{\partial \boldsymbol{z}_0} \qquad (8)$$

from an initial value

$$\frac{\partial \boldsymbol{z}(0)}{\partial \boldsymbol{z}_0} = \boldsymbol{1} \qquad (9)$$

where $\boldsymbol{1}$ is a unit matrix of $n \times n$. The existence of the Jacobi matrix indicates that the function $\boldsymbol{F}(\boldsymbol{z}_0)$ is continuously differentiable if the integral in Eq.(5) is calculated.

3. Monotoneity of Homotopy Path

3.1. Expression with real numbers

We assume that the integral in Eq.(5) can be calculated. In order to clarify the property of the complex function $\boldsymbol{F}(\boldsymbol{z})$, we express the function with a pair of real numbers:

$$F_k(\boldsymbol{z}) = F_k^r(\boldsymbol{z}^r, \boldsymbol{z}^i) + \mathrm{i}F_k^i(\boldsymbol{z}^r, \boldsymbol{z}^i) \qquad (10)$$
$$k = 1, \cdots, n, \quad \boldsymbol{z} = \boldsymbol{z}^r + \mathrm{i}\boldsymbol{z}^i$$

where $F_k^r, F_k^i \in \boldsymbol{R}$ and $\boldsymbol{z}^r, \boldsymbol{z}^i \in \boldsymbol{R}^n$. This permits us to define a new function $\hat{\boldsymbol{F}}(\hat{\boldsymbol{z}}) : \boldsymbol{R}^{2n} \mapsto \boldsymbol{R}^{2n}$ where

$$\begin{cases} \hat{F}_1 = F_1^r \\ \hat{F}_2 = F_1^i \end{cases}, \cdots, \begin{cases} \hat{F}_{2n-1} = F_n^r \\ \hat{F}_{2n} = F_n^i \end{cases}, \qquad (11)$$

$$\begin{cases} \hat{z}_1 = z_1^r \\ \hat{z}_2 = z_1^i \end{cases}, \cdots, \begin{cases} \hat{z}_{2n-1} = z_n^r \\ \hat{z}_{2n} = z_n^i \end{cases}. \qquad (12)$$

Using Cauchy-Riemann equations, we can prove that the determinant of the Jacobi matrix of the function $\hat{\boldsymbol{F}}$ satisfies the following inequality [1]:

$$\det\left[\frac{\partial \hat{\boldsymbol{F}}}{\partial \hat{\boldsymbol{z}}}\right] \geq 0. \qquad (13)$$

3.2. Newton homotopy

In order to solve the equation $\hat{\boldsymbol{F}}(\hat{\boldsymbol{z}}) = 0$ instead of Eq.(6), we use Newton homotopy method. We define the Newton homotopy function $\hat{\boldsymbol{H}} : \boldsymbol{R}^{2n} \mapsto \boldsymbol{R}^{2n+1}$ by

$$\hat{\boldsymbol{H}}(\hat{\boldsymbol{z}}, \alpha) \equiv \alpha\hat{\boldsymbol{F}}(\hat{\boldsymbol{z}}) + (1-\alpha)\left[\hat{\boldsymbol{F}}(\hat{\boldsymbol{z}}) - \hat{\boldsymbol{F}}(\hat{\boldsymbol{a}})\right] \qquad (14)$$

where $\alpha \in \boldsymbol{R}$ is a homotopy parameter and $\hat{\boldsymbol{a}} \in \boldsymbol{R}^{2n}$ is a given vector. This function satisfies equations

$$\hat{\boldsymbol{H}}(\hat{\boldsymbol{a}}, 0) = \boldsymbol{0}, \quad \hat{\boldsymbol{H}}(\hat{\boldsymbol{z}}, 1) = \hat{\boldsymbol{F}}(\hat{\boldsymbol{z}}). \qquad (15)$$

We define Newton homotoy equation by

$$\hat{\boldsymbol{H}}(\hat{\boldsymbol{z}}, \alpha) = \boldsymbol{0} \qquad (16)$$

and we define a homotopy path by

$$\hat{\boldsymbol{H}}^{-1}(\boldsymbol{0}) \equiv \left\{(\hat{\boldsymbol{z}}, \alpha) \mid \hat{\boldsymbol{H}}(\hat{\boldsymbol{z}}, \alpha) = \boldsymbol{0}\right\}. \qquad (17)$$

We trace the homotopy path from the given initial point $[\hat{\boldsymbol{a}}, 0]$ and if we arrive at $\alpha = 1$, then we have a solution $\boldsymbol{z}_0$ of Eq.(6).

3.3. Monotoneity of homotopy path

We define a vector $\hat{\boldsymbol{y}} \equiv [\hat{\boldsymbol{z}}, \alpha] \in \boldsymbol{R}^{2n+1}$. Assuming that the homotopy path is represented by $\hat{\boldsymbol{y}}(\theta)$ where $\theta \in \boldsymbol{R}$ represents the arclength of the homotopy path, the following equation

$$\hat{\boldsymbol{H}}(\hat{\boldsymbol{y}}(\theta)) = \boldsymbol{0} \qquad (18)$$

is satisfied. The homotopy path $\hat{\boldsymbol{y}}(\theta)$ satisfies the basic differential equations[1]

$$\frac{\mathrm{d}\hat{y}_k}{\mathrm{d}\theta} = (-1)^{k+1}\det\left[\frac{\partial \hat{\boldsymbol{H}}}{\partial \hat{\boldsymbol{y}}}\right]_{-k}, \quad k = 1, \cdots, 2n+1 \qquad (19)$$

where $[\cdot]_{-k}$ means that the kth column of the matrix $[\cdot]$ is removed.

Using Eq.(19) and Eq.(13), we obtain the following inequality:

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\theta} = \det\left[\frac{\partial \hat{\boldsymbol{H}}}{\partial \hat{\boldsymbol{y}}}\right]_{-(2n+1)} \qquad (20)$$
$$= \det\left[\frac{\partial \hat{\boldsymbol{F}}}{\partial \hat{\boldsymbol{z}}}\right] \qquad (21)$$
$$\geq 0. \qquad (22)$$

This inequality shows that the homotopy path cannot reverse itself in α. That is, the homotopy path is monotonic in α. As a result, if the integral in Eq.(5) is calculated on the homotopy path and the homotopy path is bounded, we always arrive at $\alpha = 1$ and obtain the solution of Eq.(6).

3.4. General homotopy

After we obtain a sufficient number of the complex periodic solutions by the Newton homotopy, we detect real periodic solutions using a general homotopy. We consider a circuit parameter μ and redefine the determining equation by

$$\hat{\boldsymbol{F}}(\hat{\boldsymbol{z}} \mid \mu) \equiv \hat{\boldsymbol{F}}(\hat{\boldsymbol{z}}) = \boldsymbol{0} \qquad (23)$$

When the circuit parameter μ is increased or decreased, the solution curve is followed by the general homotopy function $\hat{\boldsymbol{G}} : \boldsymbol{R}^{2n+1} \mapsto \boldsymbol{R}^{2n}$ defined by

$$\hat{\boldsymbol{G}}(\hat{\boldsymbol{z}}, \mu) \equiv \hat{\boldsymbol{F}}(\hat{\boldsymbol{z}} \mid \mu). \tag{24}$$

The general homotopy equation is defined by

$$\hat{\boldsymbol{G}}(\hat{\boldsymbol{z}}, \mu) = \boldsymbol{0} \tag{25}$$

and the solution path is defined by

$$\hat{\boldsymbol{G}}^{-1}(\boldsymbol{0}) \equiv \left\{ (\hat{\boldsymbol{z}}, \mu) \mid \hat{\boldsymbol{G}}(\hat{\boldsymbol{z}}, \mu) = \boldsymbol{0} \right\}. \tag{26}$$

In the same way as Eq.(20), we obtain

$$\frac{\mathrm{d}\mu}{\mathrm{d}\theta} = \det \left[\frac{\partial \hat{\boldsymbol{G}}}{\partial \hat{\boldsymbol{y}}} \right]_{-(2n+1)} \tag{27}$$

$$= \det \left[\frac{\partial \hat{\boldsymbol{F}}}{\partial \hat{\boldsymbol{z}}} \right] \tag{28}$$

$$\geq 0. \tag{29}$$

This inequality shows that the general homotopy path is also monotonic in μ. As a result, if the integral in Eq.(5) is calculated on the general homotopy path, we can trace the path without reverse itself in μ and we can detect the real periodic solutions throughout the parameter space.

4. Algorithm for Finding Real Periodic Solutions

4.1. Procedure

Based on the property of the Newton homotopy and the general homotopy, we detect the real periodic solutions in the circuit parameter space, using the following procedure:

Step 1: We give a set of circuit parameters.

Step 2: We give enough initial vectors $\hat{\boldsymbol{a}}$ and find all or almost all complex periodic solutions using the Newton homotopy. Based on the monotoneity of the path, we can find solutions if the integral of Eq.(5) is calculated and the path is bounded.

Step 3: We detect real periodic solutions using the general homotopy. Based on the monotoneity of the general homotopy, we can detect the real solutions in the parameter space.

If we obtain enough solutions in Step 2, we need not execute the Newton homotopy for other values of the circuit parameter.

4.2. Branch switching on turning point

We trace the homotopy path using a predictor-corrector method [6]. However, the trace method on the turning point (saddle-node bifurcation point) is different from the usual method. Because the homotopy path is monotonic, the homotopy path has other branches besides the usual return path. That is, if we find turning point on the homotopy path of real solutions, we have to trace new branches of complex solutions and if we find turning point on the homotopy path of complex solutions, we have to trace new branches of real solutions[7].

In order to trace the new branch, we set the predictor $\hat{\boldsymbol{y}}^*$ on the turning point of the real solution curve as

$$\hat{\boldsymbol{y}}^* = \hat{\boldsymbol{y}} + \delta \hat{\boldsymbol{l}} \tag{30}$$

$$\hat{\boldsymbol{l}} = \frac{1}{\sqrt{n}} (\underbrace{0, 1, 0, 1 \ldots, 0, 1}_{2n}, 0) \tag{31}$$

where $\hat{\boldsymbol{y}}$ is the turning point and δ is a step length. Using the predictor, we can trace the complex branch. In the same way, we set the predictor on the turning point of the complex path as

$$\hat{\boldsymbol{y}}^* = \hat{\boldsymbol{y}} + \delta \hat{\boldsymbol{l}} \tag{32}$$

$$\hat{\boldsymbol{l}} = \frac{1}{\sqrt{n}} (\underbrace{1, 0, 1, 0, \ldots, 1, 0}_{2n}, 0). \tag{33}$$

Using the predictor, we can switch the path to the real branch.

5. Application to Simple Example

We apply the proposed method to a RLC-resonance circuit shown in Fig.1. The scaled circuit equation is

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} = -U - \zeta I(\Psi) + E \sin(t) \tag{34}$$

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \eta I(\Psi). \tag{35}$$

where ζ and η corresponds to resistance and capacitive susceptance, respectively. The magnetizing characteristics of nonlinear inductor is approximated by $I(\Psi) = \Psi^5$. We detect 1/5-subharmonic solutions of the equation. That is, we set the period $T = 10\pi$.

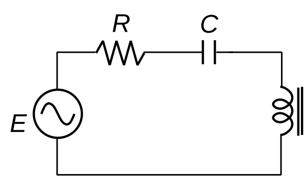

Figure 1: RLC resonance circuit with nonlinear inductor.

First, we fixed the circuit parameters $\zeta = 0.15$, $\eta = 0.4$ and $E = 0.5$. We give randomly generated 10000 initial vectors to the Newton homotopy. Because the integral of Eq.(5) is not always calculated in the interval $[0, 10\pi]$ on the homotopy path, we obtain 4792 solutions. The increase of solutions obtained by the Newton homotopy is shown in

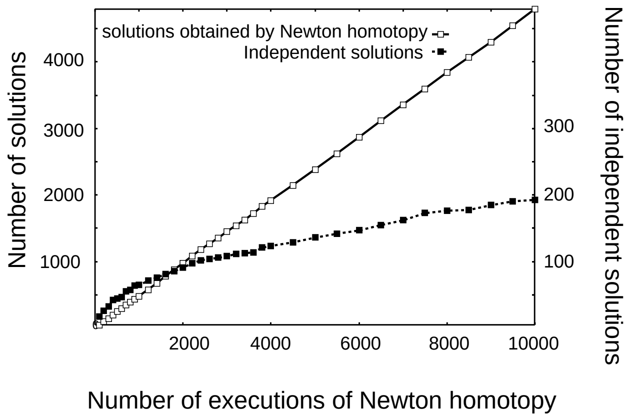

Figure 2: The number of solutions and independent solutions obtained by the Newton homotopy.

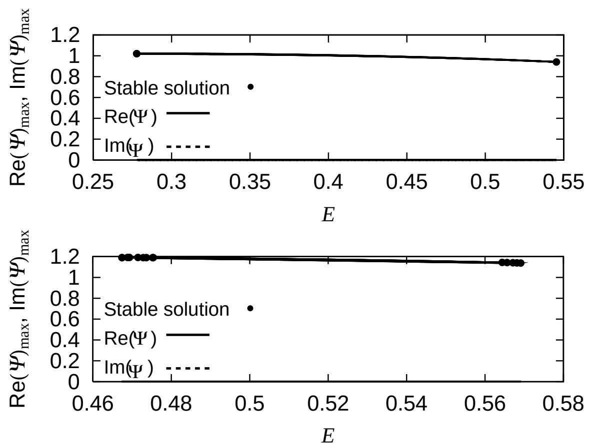

Figure 3: Real solutions calculated by the general homotopy from the real solutions obtained by the Newton homotopy. Because the solutions are real, the imaginary part Im(Ψ) is equal to 0.

Fig.2. After we remove equivalent solutions, we obtain 193 independent solutions shown in Fig.2.

We find only 4 real solutions in 193 solutions. Applying the general homotopy of the parameter E to the real solutions, we trace the real solutions. Figure 3 shows the maximal values of the real part and imaginary part of the solutions Ψ. The imaginary part is equal to 0. The 4 solutions consists of two pairs and the pair solutions become multiple root on the turning points. Although we traced complex solutions from the turning points, we cannot find other real solutions.

Applying the general homotopy to the other 189 complex solutions, we can find real solutions. Two examples are shown in Fig. 4. We can confirm that the real solutions are detected by the proposed method.

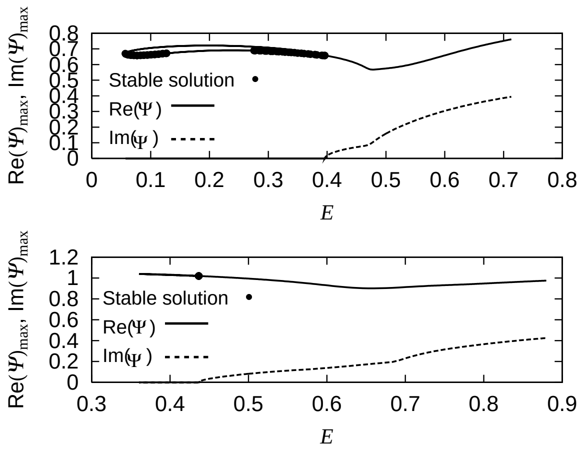

Figure 4: Examples of real solutions obtained by the general homotopy in E. The region where the maximal values of the imaginary part Im(Ψ) is equal to 0 has real solutions.

6. Conclusion

We proposed a complex homotopy method for finding periodic solutions of circuit equations. By extending the state variables to complex numbers, we have complex solutions and the monotoneity of the homotopy path. Based on the monotoneity, we can trace the homotopy path to complex periodic solutions by Newton homotopy, if the integral of Eq.(5) is calculated and the path is bounded. In general homotopy, we can trace the homotopy path to any circuit parameters if the integral is calculated. We confirmed the efficiency of the proposed method by the RLC-resonance circuit. However, the analysis of the failure in the integral of Eq.(5) is a future problem.

Acknowledgments

This work was supported by the 21st Century COE Program 14213201 and MEXT 14780211.

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