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Design of Nonlinear RLS Wiener Fixed-Point Smoother using Covariance Information in Linear Discrete-Time Stochastic Systems

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Abstract– This paper proposes the extended RLS Wiener fixed-point smoother and filter. It is assumed that the signal is observed with the nonlinear mechanism and with the additive white observation noise.

1. Introduction

The extended RLS (recursive least-squares) fixed-point smoother [1], filter [1], [2] and predictor [3] have been designed for the signal observed with the nonlinear mechanism and with additional observation noise by assuming that the autocovariance function of the signal is expressed in the semi-degenerate kernel form. The semi-degenerate kernel expressing the autocovariance function of the signal as finite sum of non-random functions seems to be appropriate for the estimation of the stochastic signal generally. However, this kind of semi-degenerate kernel expression has the limitation in representing the autocovariance function of the stochastic signal generally, for example, for the signal generated by the AR (autoregressive) model. From this reason, this paper examines to design the extended RLS Wiener fixed-point smoother and filter by considering the expression method of Φ, C and $K(s,s)$ from the autocovariance function of the signal.

2. Least-Squares Smoothing Problem for Linear Modulation

Let a scalar observation equation be given by

$$y(k)=H(k)z(k)+v(k),\ z(k)=Cx(k), \qquad (1)$$

in linear discrete-time stochastic systems, where $z(k)$ is a scalar signal, $H(k)$ is a scalar observation function, $x(k)$ is an $n\times 1$ state variable, C is a $1\times n$ a vector generating $z(k)$ from $x(k)$ and $v(k)$ is white observation noise. It is assumed that the signal and the observation noise are mutually independent and that $z(k)$ and $v(k)$ are zero mean. Let the autocovariance function of $v(k)$ be given by

$$E[v(k)v(s)]=R\delta_K(k-s),\ R>0. \qquad (2)$$

Here, $\delta_K(\cdot)$ denotes the Kronecker δ function.

Let $K(k,s)$ denote the autocovariance function of the state variable $x(k)$ and let $K(k,s)$ be expressed in the semi-degenerate kernel form as

$$K(k,s)=\begin{cases}A(k)B^T(s), & 0\le s\le k,\\ B(k)A^T(s), & 0\le k\le s,\end{cases} \qquad (3)$$

$A(k)=\Phi^k$, $B^T(s)=\Phi^{-s}K(s,s)$, where Φ represents the system matrix in the state equations for $x(k)$.

Let a fixed-point smoothing estimate $\hat{x}(k,L)$ of $x(k)$ be given by

$$\hat{x}(k,L)=\sum_{i=1}^{L}h(k,i,L)y(i) \qquad (4)$$

as a linear transformation of the observed values $\{y(i),\quad 1\le i\le L\}$, where $h(k,i,L)$ and k are referred to be an impulse response function and the fixed point respectively.

The impulse response function which minimizes the mean-square value of the fixed-point smoothing error,

$$J=E[\|x(k)-\hat{x}(k,L)\|^2], \qquad (5)$$

satisfies

$$E[x(k)y^T(s)]=\sum_{i=1}^{L}h(k,i,L)E[y(i)y^T(s)] \qquad (6)$$

by an orthogonal projection lemma [4]:

$$x(k)-\sum_{i=1}^{L}h(k,i,L)y(i)\perp y(s),\ 0\le s,k\le L. \qquad (7)$$

Here, '$\perp$' denotes the notation of the orthogonality. From (5) and (7), the impulse response function satisfies the Wiener-Hopf equation

$$E[x(k)y^T(s)]=\sum_{i=1}^{L}h(k,i,L)E[y(i)y^T(s)]. \qquad (8)$$

Substituting (1) and (2) into (8), we obtain

$$h(k,s,L)R(s)=K(k,s)C^TH^T(s)-\sum_{i=1}^{L}h(k,i,L)H(i)CK(i,s)C^TH^T(s) \qquad (9)$$

3. RLS Wiener Fixed-Point Smoothing and Filtering Algorithms

In [Theorem 1], the RLS Wiener fixed-point smoothing and filtering algorithms, using the covariance information of the signal and observation noise, for the observation equation (1) with the linear modulation, are shown.

[Theorem 1]

Let the linear observation equation for the signal $z(k)$ be given by (1). Let the autocovariance function of the state variable $x(k)$ be expressed by (3) and let the variance of white observation noise be R. Then, the RLS algorithms for the fixed-point smoothing and filtering estimates consist of the following equations.

Fixed-point smoothing estimate of $z(k)$ at the fixed point k: $\hat{z}(k,L) = C\hat{x}(k,L)$

Fixed-point smoothing estimate of $x(k)$ at the fixed point k: $\hat{x}(k,L)$

$\hat{x}(k,L) = \hat{x}(k,L-1) + h(k,L,L)(y(L) - H(L)\hat{z}(L,L-1))$

Smoother gain: $h(k,L,L)$

$h(k,L,L) = (K(k,k)(\Phi^T)^{L-k}C^T H^T(L) - q(k,L-1)\Phi^T C^T H^T(L))/$
$(R + H(L)CK(L,L)C^T H^T(L) - H(L)C\Phi S(L-1)\Phi^T C^T H^T(L))$

$q(k,L) = q(k,L-1)\Phi^T + h(k,L,L)H(L)C(K(L,L) - \Phi S(L-1)\Phi^T)$, $q(k,k) = S(k)$

Filtering estimate of $z(k)$: $\hat{z}(k,k) = H(k)C\hat{x}(k,k)$

Filtering estimate of $x(k)$: $\hat{x}(k,k)$

$\hat{x}(k,k) = \Phi\hat{x}(k-1,k-1) + G(k)(y(k) - H(k)C\Phi\hat{x}(k-1,k-1))$, $\hat{x}(0,0) = 0$

One-step ahead prediction estimate of the signal $z(k)$: $\hat{z}(k,k-1)$

$\hat{z}(k,k-1) = C\hat{x}(k,k-1)$

One-step ahead prediction estimate of the state variable $x(k)$: $\hat{x}(k,k-1)$

$\hat{x}(k,k-1) = \Phi\hat{x}(k-1,k-1)$

$S(k) = \Phi S(k-1)\Phi^T + G(k)H(k)C(K(k,k) - \Phi S(k-1)\Phi^T)$, $S(0) = 0$

Filter gain: $G(k)$

$G(k) = (K(k,k)C^T H^T(k) - \Phi S(k-1)\Phi^T C^T H^T(k))/$
$(R + H(k)CK(k,k)C^T H^T(k) - H(k)C\Phi S(k-1)\Phi^T C^T H^T(k))$

Proof. The fixed-point smoothing and filtering equations in [Theorem 1] are immediately derived by applying the estimation technique in [5], using covariance information for the conventional observation equation with additive white noise, to the case of the observation equation (1) with the linear modulation.

4. Extended RLS Wiener Fixed-Point Smoother and Filter for Nonlinear Modulation

Let a scalar observation equation with the nonlinear mechanism be given by

$$y(k) = f(z(k),k) + v(k),\ z(k) = Cx(k), \quad (10)$$

where the scalar signal $z(k)$ and the observation noise $v(k)$ have the same stochastic properties as those in section 2.

In the design of the extended estimators using the covariance information, as in the extended Kalman filter, we use the observation function $H(k) = \left.\frac{\partial f(z(k),k)}{\partial z(k)}\right|_{z(k)=\hat{z}(k,k-1)}$ in replacement of $H(k)$ in [Theorem 1]. Here, $\hat{z}(k,k-1) = C\Phi\hat{x}(k-1,k-1)$ represents the one-step ahead prediction estimate of the signal $z(k)$. Also, we replace $H(L)\hat{z}(L,L-1)$ and $H(k)C\hat{x}(k,k-1)$ in [Theorem 1] with $f(\hat{z}(L,L-1),L)$ and $f(\hat{z}(k,k-1),k)$ respectively.

Accordingly, the RLS Wiener fixed-point smoothing and filtering algorithms with the nonlinear observation mechanism are summarized in [Theorem 2].

[Theorem 2]

Let the observation equation with the nonlinear observation mechanism be given by (10). Let the autocovariance function of the state variable $x(k)$ be expressed by (3) and let the variance of white observation noise be R. Then, the extended RLS Wiener fixed-point smoothing and filtering algorithms using the covariance information of the signal and observation noise consist of the following equations.

Fixed-point smoothing estimate of the signal $z(k)$ at the fixed point k: $\hat{z}(k,L)$

$\hat{z}(k,L) = C\hat{x}(k,L)$

Fixed-point smoothing estimate of the state variable $x(k)$ at the fixed point k: $\hat{x}(k,L)$

$\hat{x}(k,L) = \hat{x}(k,L-1) + h(k,L,L)(y(L) - f(\hat{z}(L,L-1),L))$

Smoother gain: $h(k,L,L)$

$h(k,L,L) = (K(k,k)(\Phi^T)^{L-k}C^T H^T - q(k,L-1)\Phi^T C^T H^T(L))/$
$(R + H(L)CK(L,L)C^T H^T(L) - H(L)C\Phi S(L-1)\Phi^T C^T H^T(L))$

$q(k,L) = q(k,L-1)\Phi^T +$
$h(k,L,L)H(L)C(K(L,L) - \Phi S(L-1)\Phi^T)$,
$q(L,L) = S(L)$

Filtering estimate of the signal $z(k)$:

$\hat{z}(k,k) = C\hat{x}(k,k)$

Filtering estimate of the state variable $x(k)$: $\hat{x}(k,k)$

$\hat{x}(k,k) = \Phi\hat{x}(k-1,k-1) + G(k)(y(k) - f(\hat{z}(k,k-1),k))$,
$\hat{x}(0,0) = 0$

One-step ahead prediction estimate of the signal $z(k)$: $\hat{z}(k,k-1)$

$\hat{z}(k,k-1) = C\hat{x}(k,k-1)$

One-step ahead prediction estimate of the state variable $x(k)$: $\hat{x}(k,k-1)$

$$\hat{x}(k,k-1)=\Phi\hat{x}(k-1,k-1)$$

$$S(k)=\Phi S(k-1)\Phi^T+G(k)H(k)C(K(k,k)-\Phi S(k-1)\Phi^T),$$

$$S(0)=0$$

Filter gain: $G(k)$

$$G(k)=(K(k,k)C^TH^T(k)-\Phi S(k-1)\Phi^TC^TH^T(k))$$
$$/(R+H(k)CK(k,k)C^TH^T(k)-H(k)C\Phi S(k-1)\Phi^TC^TH^T(k))$$

Here, the observation function is given by

$$H(k)=\left.\frac{\partial f(z(k),k)}{\partial z(k)}\right|_{z(k)=\hat{z}(k,k-1)}.$$

5. Expression of Autocovariance function of Stochastic Signal

For the AR model of order M, the system matrix Φ, the observation vector C and the autovariance function $K(s,s)$ of the state variable $x(k)$ are expressed as follows [5].

$$\Phi=\begin{bmatrix}0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & \cdots & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 1\\ -a_M & -a_{M-1} & -a_{M-2} & \cdots & -a_2 & -a_1\end{bmatrix},$$

$$C=\begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix},$$

$$K(s,s)=\begin{bmatrix}K_z(0) & K_z(1) & K_z(2) & \cdots & K_z(M-2) & K_z(M-1)\\ K_z(1) & K_z(0) & K_z(1) & \cdots & K_z(M-3) & K_z(M-2)\\ K_z(2) & K_z(1) & K_z(0) & \cdots & K_z(M-2) & K_z(M-3)\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ K_z(M-1) & K_z(M-2) & K_z(M-3) & \cdots & K_z(0) & K_z(1)\\ K_z(M) & K_z(M-1) & K_z(M-2) & \cdots & K_z(1) & K_z(0)\end{bmatrix}. \quad (11)$$

6. A Numerical Simulation Example

Let a scalar observation equation with the nonlinear mechanism be given by

$$y(k)=f(z(k),k)+v(k),\ z(k)=Cx(k),$$
$$f(z(k),k)=\cos(2\pi f_c k\Delta+m_A z(k)),$$
$$f_c=1{,}000(Hz),\ \Delta=0.0001,\ m_A=1.2. \quad (12)$$

The nonlinear function in (12) appears in the phase modulation of analogue communication systems. Here, f_c, Δ and m_A represent the carrier frequency, the sampling period of the signal $z(k)$ and the phase sensitivity respectively. The observation function is given by

$$H(k)=\left.\frac{\partial f(z(k),k)}{\partial z(k)}\right|_{z(k)=\hat{z}(k,k-1)}=-m_A\sin(2\pi f_c k\Delta+m_A\hat{z}(k,k-1)).$$

Let the observation noise $v(k)$ be the zero-mean white Gaussian process with the variance R, $N(0,R)$.

Let the autocovariance function of the signal $z(k)$ be given by (3), where

$$\Phi=\begin{bmatrix}0 & 1\\ -a_2 & -a_1\end{bmatrix},\ C=\begin{bmatrix}1 & 0\end{bmatrix},$$

$$K(s,s)=\begin{bmatrix}K_z(0) & K_z(1)\\ K_z(1) & K_z(0)\end{bmatrix},$$

$$a_1=-0.1,\ a_2=-0.8, \quad (13)$$

If we substitute (13) into the extended estimation algorithms of [Theorem 2], we can calculate the fixed-point smoothing estimate $\hat{z}(k,L)$ at the fixed point k and the filtering estimate $\hat{z}(k,k)$ of the signal recursively.

Fig.1 illustrates the signal $z(k)$, the filtering estimate $\hat{z}(k,k)$ and the fixed-point smoothing estimate $\hat{z}(k,k+5)$ vs. k for the white Gaussian observation noise $N(0,0.5^2)$ when the expression is used. In Fig.1, the fixed-point smoothing estimate is superior in estimation accuracy to the filtering estimate. Fig.2 illustrates the mean-square values (MSVs) of the fixed-point smoothing error $z(k)-\hat{z}(k,k+Lag)$, $L=k+Lag$, and the filtering error $z(k)-\hat{z}(k,k)$ by the extended RLS Wiener fixed-point smoother in [Theorem 2] for the observation noises $N(0,0.3^2)$, $N(0,0.5^2)$, $N(0,0.7^2)$ and $N(0,1)$ vs. Lag, $0\le Lag\le 10$, when the expression method for the parameters Φ, C and $K(s,s)$ is used. For $Lag=0$, the MSV of the filtering error is evaluated. The MSVs of the fixed-point smoothing and filtering errors are evaluated by $\sum_{k=1}^{250}\sum_{i=1}^{Lag}(z(k)-\hat{z}(k,k+i))^2/(250\cdot Lag)$ and $\sum_{k=1}^{250}(z(k)-\hat{z}(k,k))^2/250$. Fig.2 shows that the estimation accuracy of the fixed-point smoother is improved in comparison with the filter. In Fig.2, as the noise variance becomes large, the estimation accuracies of the smoother and the filter are degraded.

For references, the AR model, which generates the signal process, is given by

$$z(k+1)=-a_1z(k)-a_2z(k-1)+w(k+1),$$
$$E[w(k)w(s)]=\sigma^2\delta_K(k-s). \quad (14)$$

7. Conclusions

In this paper, the extended RLS Wiener fixed-point smoother and filter have been designed in discrete-time stochastic systems. The expression methods for Φ, C and $K(s,s)$ from the autocovariance function of the signal have been used. The expression is suitable for representing the parameters Φ, C and $K(s,s)$ of general stationary stochastic signal.

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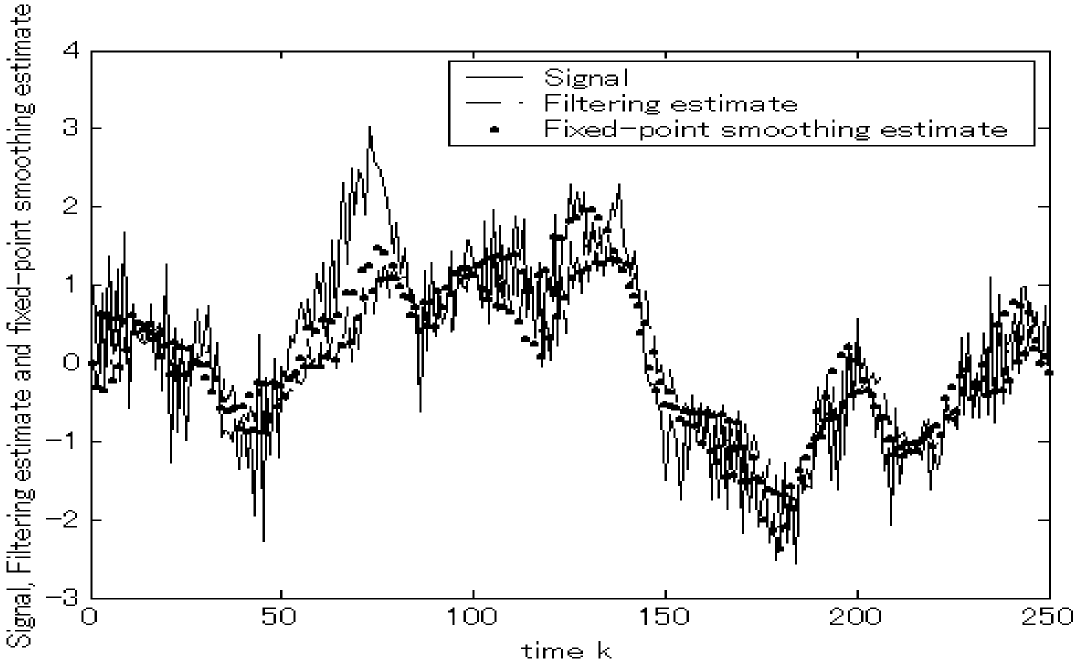

Fig.1 Filtering and fixed-point smoothing estimates vs. k.

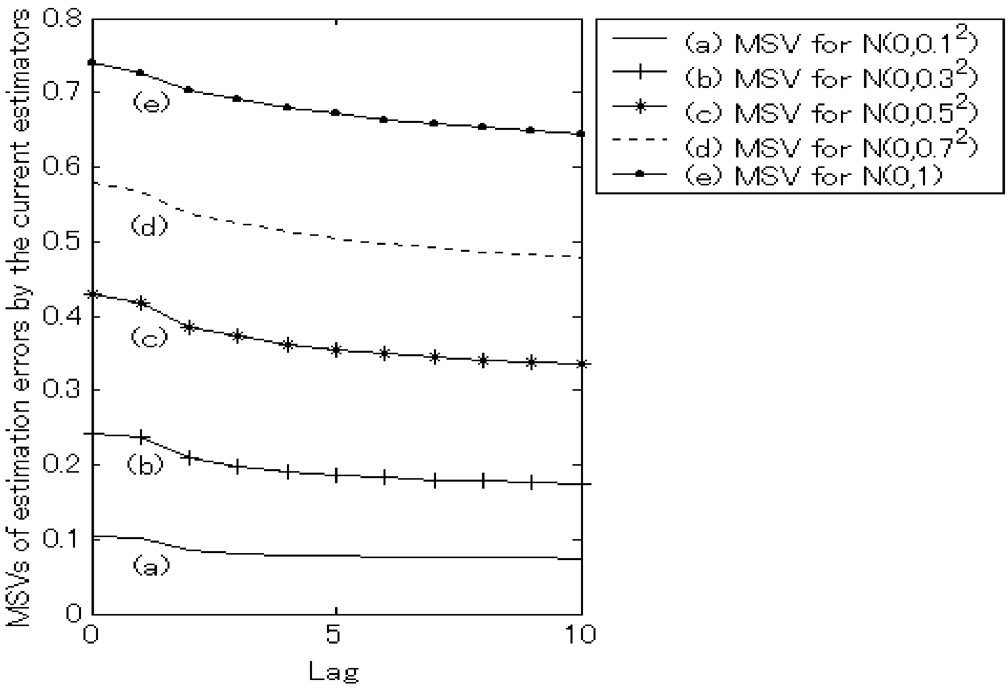

Fig.2 MSV of the estimation errors vs. Lag.