2004 International Symposium on Nonlinear
Theory and its Applications (NOLTA2004)
Fukuoka, Japan, Nov. 29 - Dec. 3, 2004

On a Stochastic Beam-Beam Interaction Model under Narrow-Band Excitation

Yong XU† Wei XU† and Gamal M.MAHMOUD ‡

† Department of Applied Mathematics, Northwestern Polytechinical University,
Xi'an, 710072, China
‡ Department of Mathematics, Faculty of Science, Assiut University,
Assiut, 71516, Egypt.
Email: hsux3@263.net, weixu@nwpu.edu.cn, gmahmoud@aun.eun.eg

Abstract– This paper is to continue our previous investigations for beam-beam interaction models in particle accelerators by considering these models under narrow-band random excitation. In this investigation the method of multiple scales and the moment methods are used to derive the equations of modulation of the amplitude and the phase, and the perturbation technique is used to seek the approximate steady state solutions. A special case is considered to illustrate this study and excellent agreement is found between analytical and numerical results. Other cases can be similarly studied. The effect of noise and detuning parameters are examined numerically and theoretically and we find that, the system demonstrates a diffused limit cycle as the intensity of noise increases. The numerical results show that multiple scale method is effective for the stochastic system.

1. Introduction

The beam-beam interaction models in particle accelerators have gained more and more attention both from a practical and a theoretical point of view [1-6]. The presence of narrow-band random process in nonlinear differential equations, which may be used to model the beam-beam interaction in physics can play an important role in nonlinear dynamical systems.

In this paper we continue our previous investigations for beam-beam interaction in particle accelerators [1-4] by considering these models under narrow-band random excitation, which may not be studied in the literature as far as we know. Some other physical and engineering models with narrow-band random excitation (or wide-band excitation), for example, Duffing oscillators, hard-spring oscillators and elastic systems, are studied in previous years [7-22]. There are many methods for studying these models, e. g., stochastic averaging method [16], multiple scale method [20,23-25], the method of equivalent linearization [6-14], quasi-static method [15,16], path integral method [17], and digital simulation [18,19,26,27].

In this study, we consider the beam-beam interaction model in particle accelerators under random narrow-band excitation of the form:

$$\ddot{x} + {\omega_0}^2 x + \varepsilon\alpha_1 g(\dot{x}) + \varepsilon\alpha_2 f(x)\xi(t) = 0 , \qquad (1)$$

where x is the displacement transverse to the ideal orbit of the particle, ω_0 and α_1 are natural frequency and small damping coefficients respectively, ε is a small parameter, α_2 denotes the intensity of nonlinear terms, $f(x)$ and $g(\dot{x})$ are generally nonlinear functions, $\xi(t)$ is a narrow-band random process and dots represent as usual differentiation with respect to time t.

Equation (1) with $\xi(t) = p(t)$ is a deterministic periodic function in t has been proposed to describe the interaction between colliding "flat" beam (one-dimensional) in intersecting storage rings [1, and references therein]. The narrow-band random process can be described as many models for instance, the stationary Gaussian random process can be represented as the stationary response of the linear filter to Gaussian white noise [28]. Here we consider the random narrow-band excitation as [29]

$$\xi(t) = h\cos(\Omega t + \gamma W(t)) , \qquad (2)$$

where $h > 0$ is deterministic amplitude of random excitation , Ω is the center frequency, $W(t)$ is a standard Wiener process, and $\gamma \geq 0$ is the bandwidth of the random excitation.

This paper is organized as follows: In Section 2, the multiple scale method is applied to derive equations of modulation of the amplitude and the phase of equation (1). As an example we consider $g(\dot{x}) = \dot{x}^5, f(x) = x^3$, $\Omega \approx 4\omega_0$ in sections 3. The 6th Runge-Kutta routine with IMSL is used to integrate equation (1) numerically. The numerical simulations are compared with the analytical results for the amplitude and excellent agreement is found. Other cases of g, f and Ω can be similarly studied. Analytical and numerical results show that the nontrivial steady state solution may change from a limit cycle to a diffused cycle as the intensity of the random excitation increases. Finally, Section 5 contains our concluding remarks.

2. The Multiple Scale Method for Equation (1)

The method of multiple scales [24] has been widely used in the analysis of deterministic systems. Rajan &

Davies [25], Nayfeh & Serhan [23] extended this method to the analysis of nonlinear systems under random external excitations. Rong et al [20] extended this method to the nonlinear systems under random parametric excitations. In this paper, the multiple scale method is used to investigate the response of system (1). Then, a uniformly approximate solution of equation (1) is sought in the form

$$x(t,\varepsilon) = x_0(T_0,T_1) + \varepsilon x_1(T_0,T_1) + \cdots, \tag{4}$$

where $T_0 = t, T_1 = \varepsilon t$ are fast and slow scale respectively.

Throughout this paper we only discuss the first-order uniform expansion of the solution $x_0(T_0,T_1)$ of equations (1). By denoting $D_0 = \frac{\partial}{\partial T_0}, D_1 = \frac{\partial}{\partial T_1}$, the ordinary-time derivatives can be transformed into partial derivatives as

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \cdots, \frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + \cdots . \tag{5}$$

Substituting equations (2), (4) and (5) into equations (1) and comparing coefficients of ε of equal powers, one obtains the following equations

$$D_0^2 x_0 + \omega_0^2 x_0 = 0, \tag{6}$$

$$\begin{aligned} D_0^2 x_1 + \omega_0^2 x_1 = &-2D_0 D_1 x_0 - 2\alpha_1 g(D_0 x_0) \\ &-\alpha_2 h f(x_0)\cos(\Omega T_0 + \gamma W(T_1)). \end{aligned} \tag{7}$$

The general solution of equation (6) can be written as

$$x_0(T_0,T_1) = a(T_1)\exp(i\omega_0 T_0) + cc, \tag{8}$$

where cc represents the complex conjugate of its preceding terms, and $a(T_1)$ is the slowly varying amplitude of the response. Substituting (8) into equations (7), one obtains

$$\begin{aligned} D_0^2 x_1 + \omega_0^2 x_1 = &-2i\omega_0 a' e^{i\omega_0 T_0} + cc - \alpha_1 g(i\omega_0 a e^{i\omega_0 T_0} \\ &+cc) - \alpha_2 h f(a e^{i\omega_0 T_0} + cc)\cos(\Omega T_0 \\ &+\gamma W(T_1)), \end{aligned} \tag{9}$$

where the prime stands for the derivative with respect to T_1 and cc represents the complex conjugate of its preceding terms. We can easily find that any particular solution of equation (9) contains secular terms generated by the first term in the right-hand side of equation (9), and, when the small-divisor terms appear depends on the form of $f(x), g(\dot{x})$ and the value of Ω. So, in next sections we will take different forms $f(x), g(\dot{x})$ and Ω to investigate the principle resonance of equation (1).

3. Equation (1) with $f(x) = x^3, g(\dot{x}) = \dot{x}^5$ and $\Omega \approx 4\omega_0$

3.1 Theoretical analysis

Let $f(x) = x^3, g(\dot{x}) = \dot{x}^5$ and $\Omega = 4\omega_0 + \varepsilon\sigma$ with σ the detuning parameter and substituting them into Eq. (9), one obtains,

$$\begin{aligned} D_0^2 x_1 + \omega_0^2 x_1 = &-2i\omega_0 a' e^{i\omega_0 T_0} - i\alpha_1 \omega_0^5 (a^5 e^{i5\omega_0 T_0} - 5a^4 \overline{a} e^{i3\omega_0 T_0} \\ &+10a^3\overline{a}^2 e^{i\omega_0 T_0}) - \frac{1}{2}\alpha_2 h(\overline{a}^3 e^{i(\omega_0 T_0 + \sigma T_1 + \gamma W(T_1))} \\ &+3\overline{a}^2 a e^{i(3\omega_0 T_0 + \sigma T_1 + \gamma W(T_1))} + \overline{a}^3 e^{-i(7\omega_0 T_0 + \sigma T_1 + \gamma W(T_1))} \\ &+3\overline{a}^2 a e^{-i(5\omega_0 T_0 + \sigma T_1 + \gamma W(T_1))}) + cc. \end{aligned} \tag{10}$$

Eliminating the secular terms in Eq. (10) yields,

$$-2i\omega_0 a' - i\alpha_1 \omega_0^5 10a^3\overline{a}^2 - \frac{1}{2}\alpha_2 h \overline{a}^3 e^{i(\sigma T_1 + \gamma W(T_1))} = 0 \tag{11}$$

The polar form of a is introduced as following,

$$a = (1/2)\rho(T_1)e^{i\theta(T_1)}. \tag{12}$$

Substituting (12) into (11) and separating the real and imaginary parts yields,

$$\rho' = -\frac{5}{16}\rho^5 \alpha_1 \omega_0^4 - \frac{1}{16\omega_0}\alpha_2 h \rho^3 \sin\varphi \tag{13a}$$

$$\rho\theta' = \frac{1}{16\omega_0}\alpha_2 h \rho^3 \cos\varphi \tag{13b}$$

where $\varphi = \sigma T_1 - 4\theta + \gamma W(T_1)$. By solving the Eqs. (13), one can obtain the first-order uniform expansion of the solution of system (1) as,

$$\begin{aligned} x_0(T_0,T_1) &= a(T_1)\exp(i\omega_0 T_0) + cc + O(\varepsilon) \\ &= \frac{1}{2}\rho(T_1)\exp[i(\omega_0 T_0 + \theta(T_1))] + cc + O(\varepsilon) \\ &= \rho(T_1)\cos(\omega_0 T_0 + \theta(T_1)) + O(\varepsilon) \end{aligned} \tag{14}$$

However, Eqs. (13) are difficult to solve exactly and the perturbation technique is carried out. Considering the assumptions: γ is sufficiently small and so γ is assumed to be zero. Now the Eqs.(13) can be written as,

$$\rho' = -\frac{5}{16}\rho^5 \alpha_1 \omega_0^4 - \frac{1}{16\omega_0}\alpha_2 h \rho^3 \sin\varphi, \tag{15a}$$

$$\rho\varphi' = \sigma\rho - \frac{4}{16\omega_0}\alpha_2 h \rho^3 \cos\varphi. \tag{15b}$$

Now assume $\rho' = \varphi' = 0$ to obtain the steady state responses, then Eqs. (15) can be rewritten as,

$$-\frac{5}{16}\rho^5 \alpha_1 \omega_0^4 - \frac{1}{16\omega_0}\alpha_2 h \rho^3 \sin\varphi = 0 \tag{16a}$$

$$\sigma\rho - \frac{4}{16\omega_0}\alpha_2 h \rho^3 \cos\varphi = 0 \tag{16b}$$

Obviously, the trivial solution of (16), $\rho = 0$ corresponds to the trivial steady state response of system (1), and non-trivial solutions of (16) correspond to steady state responses of system (1). So next we consider the non-trivial solutions of (16). By (16) one can reach,

$$Au^2 - Bu + C = 0 \tag{17}$$

where $u = \rho^4, A = \left(\frac{5\alpha_1\omega_0}{\alpha_2}\right)^2, B = h^2, C = \left(\frac{4\sigma\omega_0}{\alpha_2}\right)^2$.

If $\Delta = h^4 - \frac{1600\alpha_1^2\sigma^2\omega_0^{12}}{\alpha_2^4} \geq 0$, then Eq. (34) has two positive roots:

$$\rho_1=\left[\frac{\alpha_2^{\,2}}{50\alpha_1^{\,2}\omega_0^{\,10}}\left(h^2+\sqrt{h^4-\frac{1600\alpha_1^{\,2}\sigma^2\omega_0^{\,12}}{\alpha_2^{\,4}}}\right)\right]^{\frac{1}{4}}, \tag{18a}$$

$$\rho_2=\left[\frac{\alpha_2^{\,2}}{50\alpha_1^{\,2}\omega_0^{\,10}}\left(h^2-\sqrt{h^4-\frac{1600\alpha_1^{\,2}\sigma^2\omega_0^{\,12}}{\alpha_2^{\,4}}}\right)\right]^{\frac{1}{4}}. \tag{18b}$$

Next, we determine the effect of the noise, i.e. $\gamma\neq 0$, on the deterministic steady state motion. To this end, we let the solution of equations (32) to be in the form

$$\rho=\rho^*+l,\quad \varphi=\varphi^*+m, \tag{19}$$

where ρ^*,φ^* are given by equations (16)-(18), and l,m are perturbation terms. Substituting equations (19) into equations (15) and neglecting the nonlinear terms of l,m, one obtains the linearization of the modulation equations (15) at (ρ^*,φ^*) as,

$$l'=c_{11}l+c_{12}m,\quad m'=c_{21}l+c_{22}m+\gamma W', \tag{20}$$

where

$$c_{11}=-\frac{25}{16}\alpha_1\omega_0^{\,4}-\frac{3}{16\omega_0}\alpha_2 h\rho*^2\sin\varphi*,\quad c_{22}=-\frac{1}{4\omega_0}\alpha_2 h\sin\varphi*,$$

$$c_{12}=-\frac{1}{16\omega_0}\alpha_2 h\rho*^3\cos\varphi*,\quad c_{21}=\frac{\sigma}{\rho*}-\frac{3}{4\omega_0}\alpha_2 h\rho*\cos\varphi*\cdot$$

Eq. (20) can be written as the following *Ito* equations

$$dX=AXdT_1+BdW(T_1), \tag{21}$$

where $A=(c_{ij})_{2\times2}$, $X=(l,m)^T$, $B=(0,0,0,\gamma)^T$.

By the method like 3.1 one obtains the first and second-order steady state moment as,

$$E\rho=\rho^*,\quad E\rho^2=(\rho^*)^2+El^2. \tag{22}$$

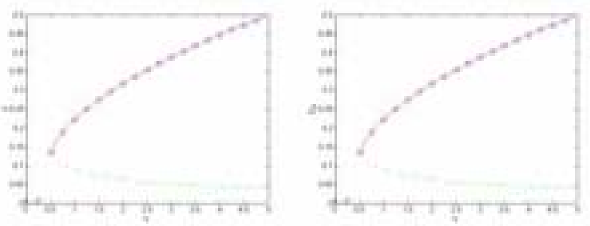
3.2. Numerical analysis

In this part we carry out the numerical analysis to verify the theoretical results in 3.1. For the method of numerical simulation, the reader is referred to Zhu [21] and Shinozuka [22,27].

By means of 6th Runge-Kutta routine (with IMSL) we integrate Eq.(1) numerically and the parameters in (1) can be selected as,

$$\alpha_1=2.0,\alpha_2=0.5,\sigma=0.001,\omega_0=1.0,\varepsilon=1.0,$$

and Ω is determined by ε,σ and ω_0. When $\gamma=0$, the variations of the steady state response and the theoretical results given by equations (18) with h are shown in Figure 1(a). As shown in Figure 1(a), the deterministic response predicted by the method of multiple scales is in good agreement with that obtained by numerical result. In Fig. 1, the horizontal axis denotes the excitation frequency h, and the vertical axis denotes the steady state responses. As $h<h^*\approx 0.566$ the steady state responses are always zero, otherwise the steady state responses increase with the increasing value of the excitation amplitude h. Also, one can see that, there are two steady state responses ρ_1,ρ_2 theoretically, but by numerical simulation we find only ρ_1 can be realized, namely, only ρ_1 is a stable steady state response and ρ_2 is an unstable one.

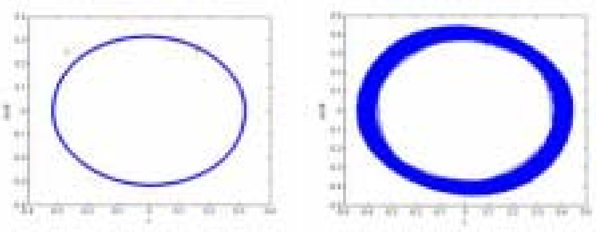


(a) $\gamma=0$ (b) $\gamma=0.1$

Figure 1 Curves of response; — theoretical solution; — ——unstable solution ; ○ ○ ○ numerical solution.
($x(0)=1.0,\dot{x}(0)=0.4$)

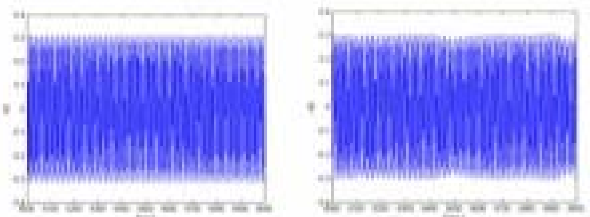
In Figure 2-3, we present some figures to describe the phase plot (where the horizontal axis denotes the displacement x(t), the vertical axis denotes the velocity dx(t)/dt) and time history (where the horizontal axis denotes the time t, the vertical axis denotes the displacement x(t)) of $x(t)$ with $\gamma=0$ and $\gamma\neq0$ with $h=2.0,x(0)=1.0,\dot{x}(0)=0.4$. One can observe that, the random noise $\gamma W(t)$ will change the steady state response of system (1) from a limit cycle to a diffused limit cycle. Further numerical simulation shows that when the intensity of the random excitation increases, the width of the diffused limit cycle will increase.

When $\gamma=0.1$, Figure 1(b) gives the variations of steady state response of system (1) numerically and analytically, and good agreement can be found immediately.



(a) $\gamma=0$ (b) $\gamma=0.1$

Figure 2 Phase Plot



(a) $\gamma=0$ (b) $\gamma=0.1$

Figure 3 Time history

4. Concluding Remarks

The approximate methods in deterministic systems such as multiple scale method, equivalent linearization method and averaging method can be extended to random systems

[20,23,25,26]. However, exact solutions of nonlinear system under random excitation are only available for a very limited number of problems, even if for the single degree of freedom nonlinear deterministic system.

In the present paper, we investigate a beam-beam interaction model under narrow-band random excitation using the method of multiple time scale. This study is considered as a continuation of our previous investigations for these models in particle accelerators.

In the theoretical analysis, one can find there are two theoretical solutions of steady state responses, numerical simulation shows that only ρ_1 is stable and can be realized, that is to say, ρ_2 is unstable and can not be realized (see Fig. (1)). We also find that the proposed system changes from a limit cycle to a diffused limit cycle as the intensity of random noise γ increases. The numerical simulations by 6^{th} Runge-Kutta routine with IMSL is carried out to verify the theoretical results and good agreement can be found.

Acknowledgements
The authors are grateful to the National Natural Science Foundation of China (Grant no. 10332030), and the Doctorate Creation Foundation of Northwestern Polytechnical University.

References

[1] T Bountis, G. M. Mahmoud, Synchronized periodic orbits in beam-beam interaction models of one and two spatial dimensions. Particle Acc. 22 ,1987, 124-147.
[2] G M Mahmoud, Stability regions for coupled Hill's equations, Physica A 242 ,1997, 239-249.
[3] M Month, J.C. Herrera (Eds.), Nonlinear dynamics and the beam-Beam Interaction. AIP Conference proceedings, No. 57, AIP, New York, 1979.
[4] W V Wedig, Dynamic stability of beams under axial forces: Lyapunov exponents for general fluctuating loads, in: W. B. Kratzig, et al., (Ess), Structural Dynamics, Balkema, Rotterdam ,1990, 141-148.
[5] G M Mahmoud, Beam-beam interaction models with a small stochastic perturbation. Physica A 216 ,1995, 445-451.
[6] Y Xu, W Xu, G Mahmoud, On a complex beam-beam interaction model with random forcing. Physica A 336 (2004), 347-360.
[7] R H Lyon, M Heckl and C B Hazelgrove, Response of hard-spring oscillator to narrow-band excitation. Journal of the Acoustical Society of America 33, 1961, 1404-1411.
[8] M F Dimentberg, Oscillations of a system with a nonlinear cubic characteristic under narrow-band excitation. Mekhanica Tveidogo Tela 6, 1970, 150-154.
[9] R N Iyengar, Random vibration of a second order nonlinear elastic system. Journal Sound and Vibration 40, 1975, 155-165.
[10] K Richard and G V Anand, Nonlinear resonance in strings under narrow band random excitation, part I: planar response and stability. Journal of Sound and Vibration 86, 1983, 85-98.
[11] H G Davies and D Nandall, Phase plane for narrow band random excitation of a Duffing oscillator. Journal of Sound and Vibration 104, 1986, 277-283.
[12] R N Iyengar, Stochastic response and stability of the Duffing oscillator under narrow-band excitation. Journal of Sound and Vibration 126, 1988, 255-263.
[13] R N Iyengar, Response of nonlinear systems to narrow-band excitation, Structural Safety 6, 1989, 177-185.
[14] W C Lennox and Y C Kuak, Narrow band excitation of a nonlinear oscillator, Journal of Applied Mechanics 43, 1976, 340-344.
[15] M Grigoriu, Probabilistic analysis of response of Duffing oscillator to narrow band stationary Gaussian excitation, Proceedings of First Pan-American Congress of Applied Mechanics, Rio de Janeiro, Brazil, 1989.
[16] H G Davis and Q Liu, The response probability density function of a Duffing oscillator with random narrow band excitation, Journal of Sound and Vibration 139, 1990, 1-8.
[17] T Kapitaniak, Stochastic response with bifurcations to non-linear Duffing's oscillator, Journal of Sound and Vibration 102, 1985, 440-441.
[18] T Fang and E H Dowell, Numerical simulations of jump phenomena in stable Duffing systems, International Journal of Nonlinear Mechanics 22, 1987, 267-274.
[19] W Q Zhu, M Lu and Q T Wu, Stochastic jump and bifurcation of a Duffing oscillator under narrow-band excitation, Journal of Sound and Vibration 165(2), 1993, 285-304.
[20] H Rong, W Xu and T Fang, Principal response of Duffing oscillator to combined deterministic and narrow-band random parametric excitation, Journal of Sound and Vibration 210(4), 1998, 483-515.
[21] W Q Zhu, Random vibration, Beijing: Science Press, 1992.
[22] M Shinozuka, Simulation of multivariate and multidimensional random processes, Journal of Sound and Vibration 19, 1971, 357-367.
[23] A H Nayfeh and S J Serhan, Response statistics of nonlinear systems to combined deterministic and random excitations, International Journal of Nonlinear Mechanics 25(5), 1990, 493-509.
[24] A H Nayfeh, Introduction to Perturbation Techniques, New York: Wiley, 1981.
[25] S Rajan and H G Davies, Multiple time scaling of the response of a Duffing oscillator to narrow-band excitations, Journal of Sound and Vibration 123, 1988, 497-506.
[26] W Xu, Q H, H Rong, T Fang, One-to two internal resonance in two-degree-of-freedom nonlinear system with narrow-band excitation, Nonliear Dyn. 27, 2002, 385-395.
[27] M Shinozuka, Digital simulation of random processes and its applications, Journal of Sound and Vibration 25, 1972, 111-128.
[28] R L Stratonovich *Topics in the Theory of Random Noise*. New-York: Gordon and Breach, 1963.
[29] W V Wedig, Invariant measures and Lyapunov exponents for generalized parameter fluctuations, Structural Safety 8, 1990, 13-25.