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Mixing Property for Markov Chains Generating Spreading Sequences

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Abstract—We consider M-phase spreading sequences of Markov chains ($M = 3, 4, \cdots$) and show that the correlational properties of M-phase spreading sequences of Markov chains are generally independent of the mixing property for the associated M-state Markov chains. We also give a necessary and sufficient condition for these two properties to be independent.

1. Introduction

Designing spreading sequences is one of the essential elements of spread spectrum techniques. Pursley defined the average interference parameter (AIP) and gave the average signal-to-noise ratio (SNR) at the receiver output by using Gaussian distributions whose variance was the AIP for asynchronous SSMA communication systems using binary spreading sequences [1]. This is so called the standard Gaussian approximation (SGA) with the AIP.

It is natural to consider the problem to find spreading sequences that minimize the AIP. The AIP is a function of aperiodic auto-correlation functions for spreading sequences. It is the variance of multiple access interference (MAI) regarding the data symbols, the time delays, and the phase shifts as random variables for fixed spreading sequences. In [2], an N-dimensional random vector $(X_1, X_2, \cdots, X_N)$ taking values in $\mathcal{S}^N$ is considered, where $\mathcal{S} = \left\{1, e^{\frac{1}{M}2\pi\sqrt{-1}}, e^{\frac{2}{M}2\pi\sqrt{-1}}, \cdots, e^{\frac{M-1}{M}2\pi\sqrt{-1}}\right\}$. For such a vector, like a stationary process, requiring $Cov(X_1, X_{1+\ell}) = Cov(X_n, X_{n+\ell})$ $(n = 1, 2, \cdots, N-\ell)$ for $\ell = 0, 1, 2, \cdots, N-1$, the expected value of the AIP for complex-valued sequences becomes a function over $\boldsymbol{R}^{2N}$ with respect to $\Re(Cov(X_1, X_{1+\ell}))$ and $\Im(Cov(X_1, X_{1+\ell}))$, where $Cov(X, Y)$ denotes the covariance of random variables X and Y. $\Re(\alpha)$ and $\Im(\alpha)$ denote respectively the real part and imaginary part of the complex number α. Then it is shown that the AIP is a positive definite quadratic form over $\boldsymbol{R}^{2N}$ and global minimizers are given by $\Re(Cov(X_1, X_{1+\ell})) = ar^\ell + b/r^\ell$, and $\Im(Cov(X_1, X_{1+\ell})) \equiv 0$ with $r = -2 + \sqrt{3}$, where a and b are constants determined by the property of aperiodic auto-correlation functions $C(\ell)$.

To construct spreading sequences with the decay rate r of correlational property, spreading sequences generated by PL (piecewise linear) Markov maps with the mixing rate r were proposed in [3].

As far as symbolic dynamics of PL Markov transformations are concerned, we know that their statistical properties are equivalent to those of Markov chains [4].

In this study, we consider M-phase spreading sequences of Markov chains and show that the correlational properties of M-phase spreading sequences of Markov chains are generally independent of the mixing property for the associated M-state Markov chains. We also give a necessary and sufficient condition for these two properties to be independent. These results raise an important question on the designing method for spreading sequences in [2].

2. Spreading Sequences Generated by Markov Chains Characterized by Circulant Matrices

We know that circulant matrices can be diagonalized by Fourier matrices (see Theorem 3.2.2 in [5]). By virtue of this theorem, we can explicitly obtain the distributions of the real part of the normalized MAI for SSMA communication systems using M-phase spreading sequences of Markov chains characterized by circulant matrices as follows. Practically, this is very useful for applying these sequences to SSMA communication systems because bit error probabilities in such systems caused by the real part of MAI from other $J-1$ channels can be explicitly evaluated in advance.

Firstly we consider M-phase spreading sequences of Markov chains. Let $\boldsymbol{X} = (X_n)_{n=0}^\infty$ and $\boldsymbol{Y} = (Y_n)_{n=0}^\infty$ be Markov chains, on a state space

$$\mathcal{S} = \left\{1, e^{\frac{1}{M}2\pi\sqrt{-1}}, e^{\frac{2}{M}2\pi\sqrt{-1}}, \cdots, e^{\frac{M-1}{M}2\pi\sqrt{-1}}\right\}. \quad (1)$$

We set

$$p_{ij} = \text{Prob}\left\{X_{n+1} = e^{\frac{j-1}{M}2\pi\sqrt{-1}} \middle| X_n = e^{\frac{i-1}{M}2\pi\sqrt{-1}}\right\} = \text{Prob}\left\{Y_{n+1} = e^{\frac{j-1}{M}2\pi\sqrt{-1}} \middle| Y_n = e^{\frac{i-1}{M}2\pi\sqrt{-1}}\right\} \quad (2)$$

for $i, j = 1, 2, \cdots, M$. We define the transition matrix by $P = (p_{ij})_{i,j\in\mathcal{S}}$.

Suppose that $\boldsymbol{X}$ and $\boldsymbol{Y}$ are stationary Markov chains and mutually independent. Let their stationary distributions be

$$\text{Prob}\left\{X_n = e^{\frac{i-1}{M}2\pi\sqrt{-1}}\right\} = \text{Prob}\left\{Y_n = e^{\frac{i-1}{M}2\pi\sqrt{-1}}\right\} = 1/M \quad (3)$$

for $i = 1, 2, \cdots, M$. Then we have

$$E[X_n] = E[Y_n] = 0, \quad (4)$$

$$E\left[X_n\overline{Y_{n+\ell}}\right] = E[X_n]E\left[\overline{Y_n}\right] = 0 \quad \text{for } \ell \geq 0, \quad (5)$$

where $E[Z]$ denotes the expected value of random variable Z.

We note here that the transition matrix P is a doubly stochastic matrix by the condition (3).

Consider a class of the Markov chains stated above whose transition matrix P is circulant and set

$$P = \begin{pmatrix} 1-\sum_{i=1}^{M-1} p_i & p_1 & \cdots & p_{M-1} \\ p_{M-1} & 1-\sum_{i=1}^{M-1} p_i & \cdots & p_{M-2} \\ \vdots & \vdots & \vdots & \vdots \\ p_1 & p_2 & \cdots & 1-\sum_{i=1}^{M-1} p_i \end{pmatrix}, \quad (6)$$

where $0 \leq p_i \leq 1$ $(i = 1, 2, \cdots, M-1)$, then we have

Lemma 1

$$E\left[X_n\overline{X_{n+\ell}}\right] = E\left[Y_n\overline{Y_{n+\ell}}\right] = \lambda^\ell \quad \textit{for } \ell \geq 0, \quad (7)$$

which immediately leads to

$$\left|E\left[X_n\overline{X_{n+\ell}}\right]\right|^2 = \left|E\left[Y_n\overline{Y_{n+\ell}}\right]\right|^2 = |\lambda|^{2\ell} \quad \textit{for } \ell \geq 0. \quad (8)$$

Besides we have

Lemma 2

$$E[X_nX_{n+\ell}] = E[Y_nY_{n+\ell}] = 0 \quad \textit{for } \ell \geq 0, \quad (9)$$

which immediately leads to

$$|E[X_nX_{n+\ell}]|^2 = |E[Y_nY_{n+\ell}]|^2 = 0 \quad \textit{for } \ell \geq 0, \quad (10)$$

where $\lambda = 1 - \sum_{i=1}^{M-1} p_i + p_1 e^{\frac{M-1}{M}2\pi\sqrt{-1}} + p_2 e^{\frac{2(M-1)}{M}2\pi\sqrt{-1}} + \cdots + p_{M-1} e^{\frac{(M-1)^2}{M}2\pi\sqrt{-1}}$.

Applying Theorem 2 in [6] in conjunction with Lemma 2, we have

Corollary 1 *Consider SSMA communication systems using M-phase spreading sequences of Markov chains characterized by transition matrix P. For fixed the fractional part of the relative time delay τ, the distributions of the real part of the normalized MAI for such systems are Gaussian if P is a circulant matrix.*

Now we are in a position to obtain the distributions of the real part of the normalized MAI for SSMA communication systems using M-phase spreading sequences of Markov chains characterized by circulant matrices.

If we regard τ as the outcome of random variable T with uniform distribution on $[0, 1)$, then we have

Lemma 3 *Consider SSMA communication systems using M-phase spreading sequences of Markov chains with circulant transition matrix. The density of distribution of the real part of the normalized MAI for such systems is given by*

$$\int_0^1 \frac{1}{\sqrt{2\pi\sigma(\tau)^2}} \exp\left(-\frac{z^2}{2\sigma(\tau)^2}\right) d\tau, \quad z \in \boldsymbol{R}, \quad (11)$$

where

$$\sigma(\tau)^2 = \{(1-\tau)^2 + \tau^2\}\frac{1}{2}\left(1 + \frac{2|\lambda|^2}{1-|\lambda|^2}\right) + 2(1-\tau)\tau\frac{1}{2}(\lambda + \overline{\lambda})\left(1 + \frac{|\lambda|^2}{1-|\lambda|^2}\right). \quad (12)$$

As a first approximation, we simply follow the conventional optimization in [7]–[8]. Its reliability is experimentally discussed in [9]. The expected value of the variance $\sigma(T)^2$ is given by

$$E[\sigma(T)^2] = \frac{1}{3}\left(1 + \frac{2|\lambda|^2}{1-|\lambda|^2}\right) + \frac{1}{3}\cdot\frac{1}{2}(\lambda + \overline{\lambda})\left(1 + \frac{|\lambda|^2}{1-|\lambda|^2}\right). \quad (13)$$

Since $E[\sigma(T)^2]$ is a function of λ, we denote it by $\sigma(\lambda)^2$. We obtain

Theorem 1 *σ^2 takes the minimum value $1/(2\sqrt{3})$ if and only if $\lambda = -2 + \sqrt{3}$.*

Interestingly we encounter this very number $-2+\sqrt{3}$ again as we found in [2] and [7].

Although the optimization method in [2] does not tell us how to design the optimum spreading sequences based on the AIP, the optimization method proposed in this study explicitly gives the optimum 3-phase and Q-phase spreading sequences of Markov chains based on the CLT as follows.

Example 1 *In a class of Markov chains whose transition matrix P is circulant, the optimum 3-phase spreading sequences of Markov chains are uniquely characterized by*

$$P = \begin{pmatrix} 1-2a & a & a \\ a & 1-2a & a \\ a & a & 1-2a \end{pmatrix} \quad (14)$$

with $a = 1 - 1/\sqrt{3}$. The set of all eigenvalues of P is given by $\Lambda = \{1, -2+\sqrt{3}, -2+\sqrt{3}\}$.

Example 2 *In a class of Markov chains whose transition matrix P is circulant, the optimum Q-phase spreading sequences of Markov chains are characterized by*

$$P = \begin{pmatrix} 1-2a-c & a & a & c \\ c & 1-2a-c & a & a \\ a & c & 1-2a-c & a \\ a & a & c & 1-2a-c \end{pmatrix} \quad (15)$$

with $a+c = (3-\sqrt{3})/2$, $0 \le a \le 1$, $0 \le c \le 1$, and $0 \le 2a+c \le 1$. The set of all eigenvalues of P is given by $\Lambda = \{1, -2+\sqrt{3}, -2a-(1-\sqrt{3})/2 \pm \sqrt{-1}\{2a -(3-\sqrt{3})/2\}\}$.

3. Mixing Property of Optimum Q-Phase Spreading Sequences of Markov Chains

Let us here examine the magnitude of the second largest eigenvalue in absolute value of (15). We write the one of complex conjugate eigenvalues of (15) by

$$\rho(a) = -2a - (1-\sqrt{3})/2 + \sqrt{-1}\{2a - (3-\sqrt{3})/2\},$$

where $0 \le a \le (\sqrt{3}-1)/2$. Simple computation leads to

$$|-2+\sqrt{3}|/\sqrt{2} \le |\rho(a)| = |\overline{\rho(a)}| \le \sqrt{3}-1,$$

and $|\rho((\sqrt{3}-1)/4)| = |\rho((3-\sqrt{3})/4)| = |-2+\sqrt{3}|$. Hence we obtain

$$|-2+\sqrt{3}| < |\rho(a)| = |\overline{\rho(a)}|$$

for $0 \le a < (\sqrt{3}-1)/4$, $(3-\sqrt{3})/4 < a \le (\sqrt{3}-1)/2$, which leads to

Remark 1 *Let P be the transition matrix of Markov chain generating the optimum M-phase spreading sequences that minimizes bit error probabilities in asynchronous SSMA communication systems. The magnitude of the second largest eigenvalue in absolute value of P is not always $|-2+\sqrt{3}|$.*

Interestingly this implies that the correlational properties of M-phase spreading sequences of Markov chains are generally independent of the mixing property for the associated M-state Markov chains.

4. Mixing Property for Markov Chains and Correlational Property of Spreading Sequences of Markov Chains

Let $\mathcal{S} = \{1, 2, \cdots, M\}$ $(M \ge 2)$. We denote its power set by $\mathfrak{P}(\mathcal{S})$. We consider the measurable space $(\mathcal{S}, \mathfrak{P}(\mathcal{S}))$ and its direct product space $(X, \mathcal{B}) = \prod_0^\infty (\mathcal{S}, \mathfrak{P}(\mathcal{S}))$. Let $T : X \to X$ denoted the shift transformation defined by $T(x_0 x_1 \cdots x_n \cdots) = x_1 x_2 \cdots x_{n+1} \cdots$ for $x = x_0 x_1 \cdots x_n \cdots \in X$.

A *word* (or *block*) over $\mathcal{S}$ is a finite binary sequence of numbers from $\mathcal{S}$. A word of length n is called an *n-word*. We denote the set of all n-words over $\mathcal{S}$ by $\mathcal{S}^n$.

We are given a probability vector $\boldsymbol{p} = (p_1, \cdots, p_M)$ $(\sum_{i=1}^M p_i = 1)$ and a stochastic matrix $P = (p_{i,j})_{i,j \in \mathcal{S}}$ $(p_{i,j} \ge 0, \sum_{j=1}^M p_{i,j} = 1)$ such that $\boldsymbol{p}P = \boldsymbol{p}$.

For an n-word $a = a_1 \cdots a_n$, we define m by

$$m(\{x \in X : x_q = a_0, \cdots, x_{q+n} = a_n\}) = p_0 \prod_{k=0}^{n-1} p_{k,k+1}. \quad (16)$$

Thus m can be extended to a probability measure on $(X, \mathcal{B})$ and T preserves the measure m. We call this measure-preserving transformation the $(\boldsymbol{p}, P)$-Markov shift.

For simplicity, we suppose $\boldsymbol{p} = \frac{1}{M}(1, \cdots, 1)$ from now on.

For the $(\boldsymbol{p}, P)$-Markov shift, it is well known that the following are equivalent: (i) P is irreducible and aperiodic, (ii) T is weak-mixing, and (iii) T is strong-mixing. This implies for any $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{C}^M$,

$$\boldsymbol{u}P^n\boldsymbol{v}^* = \boldsymbol{u} \cdot \boldsymbol{v}^* + \mathcal{O}(|\lambda|^n), \quad (17)$$

where $\boldsymbol{v}^*$ is the conjugate transpose of $\boldsymbol{v}$ and λ is the second largest eigenvalue in absolute value.

On the other hand, we are now interested in not all the complex-valued M-dimensional vectors, but the specific M-dimensional vector $\boldsymbol{s} = \left(1, e^{\frac{1}{M}2\pi\sqrt{-1}}, e^{\frac{2}{M}2\pi\sqrt{-1}}, \cdots, e^{\frac{M-1}{M}2\pi\sqrt{-1}}\right)$. We obtain

$$E[X_n \overline{X_{n+\ell}}] = \boldsymbol{s}P^\ell \boldsymbol{s}^* \quad (18)$$

$$= \sum_{i=1}^{M-1} c_i \lambda_i^\ell, \quad (19)$$

where c_i is a constant, and $|\lambda| = |\lambda_1| = \cdots = |\lambda_q| > |\lambda_{q+1}| \ge \cdots \ge |\lambda_{M-1}|$ $(1 \le q \le M-1)$.

By the elements of matrix theory, we obtain

Theorem 2 *In the above-mentioned situation, let $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$ be respectively the left and right eigenvectors of P corresponding to λ_i $(1 \le i \le q)$. The decay rate of the correlation value (18) is independent of the mixing rate of the $(\boldsymbol{p}, P)$-Markov shift if and only if $\boldsymbol{s} \cdot \boldsymbol{u_i} = 0$ or $\boldsymbol{v_i} \cdot \boldsymbol{s}^* = 0$ for all i.*

5. Conclusion

In this report, we considered M-phase spreading sequences of Markov chains and showed that the correlational properties of M-phase spreading sequences of Markov chains are generally independent of the mixing property for the associated M-state Markov chains. We also gave a necessary and sufficient condition for these two properties to be independent.

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