

## Nonlinear Dynamics of the RL-Varactor Circuit in the Depletion Region

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**Abstract**—A driven nonlinear series circuit consisting of a resistor, an inductor and a diode, biased so as to operate only in the depletion region, is investigated. Rigorous phase plane analysis establishes solution behaviour, a blow up criterion, and, under certain constraints, an absorbing set which contains all bounded periodic solutions. Subharmonic solutions of various orders have also been found by computer simulation.

### 1. Introduction

Considered in this paper is the driven series resistor-inductor (RL) diode circuit shown schematically in figure 1. The resistor and the inductor are both assumed to be ideal, linear components, and the diode is modelled as a

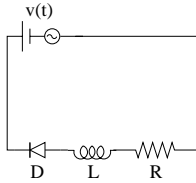


Figure 1: RL-diode circuit.

nonlinear capacitor storing charge  $Q$ . This circuit is defined by the second order differential equation

$$L\ddot{Q} + R\dot{Q} + V(Q) = v(t),$$

where  $V(Q)$  is the potential across the diode,  $v(t)$  is the driving voltage,  $L$  and  $R$  are constants which represent inductance and resistance respectively, and differentiation is with respect to time. The RL-diode circuit has been studied by electronic engineers as an example of a simple circuit which can behave chaotically. It was first investigated in 1981 by Linsay [1], who modelled the circuit and found the dynamics exhibited included period doubling and chaotic behaviour which agreed with experiment. The circuit has since been reviewed by Testa, Pérez and Jeffries [2], Azzouz, Duhr and Hasler [3, 4], Matsumoto [5] and Hasler [6], who have all tried to simplify the mathematical model and explain the chaotic behaviour. All these models have only considered the case where the voltage across the diode changes sign, which results in both diffusion and depletion capacitance effects. What has not been

considered to date is the case where the diode voltage is always negative, so that only the (weakly nonlinear) depletion capacitance applies. In this case the diode voltage is given by [7]

$$V(Q) = \phi_B \left[ 1 - \left( \frac{(m-1)Q}{\phi_B C_{j0}} + 1 \right)^{\frac{1}{1-m}} \right],$$

where  $\phi_B$  is the junction potential,  $C_{j0}$  is the zero bias junction capacitance and  $m$  is a grading coefficient. The driving voltage is taken to be  $v(t) = V_0 + V_1 \sin(\omega t)$ , where  $V_0$ ,  $V_1$  and  $\omega$  are constants. By using the following

$$Q = \frac{\phi_B C_{j0}}{m-1} \left( \left( \frac{\omega^2 L C_{j0}}{1-m} \right)^{\frac{1}{m}-1} X - 1 \right),$$

this model can be rescaled into the 4-parameter dynamical system [8]

$$X'' + \gamma X' + X^\mu = \alpha - \beta \sin \tau, \quad (1)$$

where differentiation is with respect to  $\tau$ , and  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\mu$  are positive constants given by

$$\alpha = -\frac{(V_0 - \phi_B)}{\phi_B} \left( \frac{1-m}{\omega^2 L C_{j0}} \right)^{\frac{1}{m}}, \quad \gamma = \frac{R}{\omega L},$$

$$\beta = \frac{V_1}{\phi_B} \left( \frac{1-m}{\omega^2 L C_{j0}} \right)^{\frac{1}{m}}, \quad \mu = \frac{1}{1-m}.$$

Typically  $1.5 < \mu < 2.5$ , and for this model to be valid  $X \geq 0$ . It should be noted that specifically when  $\mu = 2$ , similar mathematical models of (1) have been found in other research areas, in particular the study of the mechanics of ship capsizing [10] and the reduction of the famous Korteweg-De Vries equation into the form of (1) [11]. The dynamics of this seemingly simple model are non-trivial as it possesses subharmonic solutions.

### 2. Phase Plane Analysis

The solutions in the phase plane of the time-dependent system are now considered. Equation (1) is converted into two coupled first order differential equations

$$\begin{cases} X' = Y, \\ Y' = f(\tau) - X^\mu - \gamma Y, \end{cases} \quad (2)$$

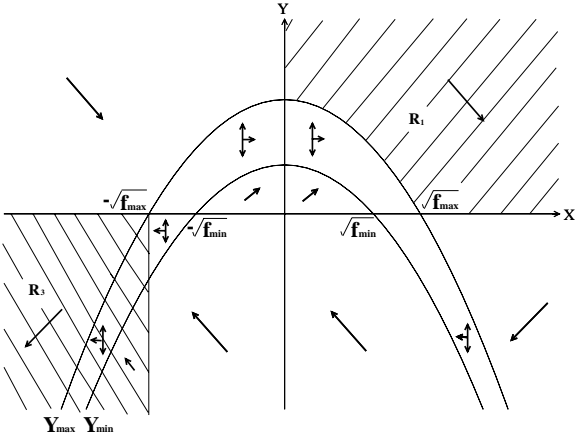


Figure 2: The curves  $Y_{min}$  and  $Y_{max}$  with the arrows showing the direction of the flow of (1) in each region.  $R_3$  denotes the blow up region.  $R_1$  denotes solutions which must cross the  $X$  axis.

where  $f(\tau) = \alpha - \beta \sin \tau$  with  $\alpha > \beta > 0$ . The function  $f(\tau)$  must satisfy

$$\alpha - \beta \leq f(\tau) \leq \alpha + \beta,$$

where  $\alpha - \beta = f_{min}$  and likewise  $\alpha + \beta = f_{max}$ . Since  $\alpha > \beta > 0$ ,  $f_{max} > f_{min} > 0$ .

The direction of motion in the phase plane is established by finding the sign of  $X'$  and  $Y'$ . Clearly  $X' = Y$  will be positive in the upper half of the phase plane, and negative in the lower half. Consider now the sign of  $Y'$ . Rearranging the  $Y'$  equation of (2) results in

$$\frac{Y'}{\gamma} + Y = \frac{f(\tau) - X^\mu}{\gamma}. \quad (3)$$

Now if  $Y'$  is to be zero, substituting  $f_{max}$  and  $f_{min}$  into (3) results in the two curves

$$\begin{cases} Y_{max} = \frac{f_{max} - X^\mu}{\gamma}, \\ Y_{min} = \frac{f_{min} - X^\mu}{\gamma}. \end{cases}$$

For the remainder of this section,  $\mu$  is taken to have the value 2. If  $\mu = 2$  then these two curves are real for all  $X$ , and are shown in figure 2, along with the direction of the flow. Only in the region between these two parabolas is the sign of  $Y'$  indeterminate, and only here can  $Y' = 0$ .

### 2.1. Blow Up Criterion

It can be shown that any solution entering region  $R_3$  of figure 2, which is defined as the region satisfying  $Y < 0$  and  $X \leq -\sqrt{f_{max}}$ , must blow up to infinity. This can be proved by considering what is known about the flow from figure 2, and deducing that this is the only possible solution of

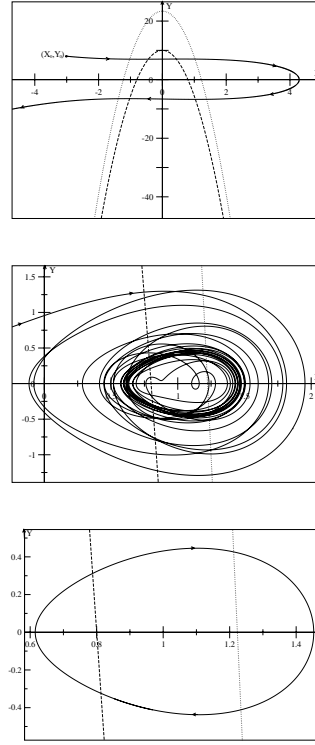


Figure 3: Solutions of (1) with  $\alpha = 1.069$ ,  $\beta = 0.428$ ,  $\gamma = 0.064$  and  $\mu = 2$ . Dashed line refers to  $Y_{min}$ . Dotted line refers to  $Y_{max}$ . Top: Solution blowing up. Middle: Period 1 limit cycle with transient. Bottom: Same period 1 limit cycle without transient.

the system [8]. This criterion is important as it allows for example, numerical searches for periodic solutions to be terminated as soon as a solution is detected to be blowing up.

### 2.2. Crossing the X Axis

A possible solution behaviour is that from any point in region  $R_1$  of figure 2, the solution could approach the  $X$  axis as  $\tau$  goes to infinity and never actually cross it, as the fact that  $X' > 0$  and  $Y' < 0$  in  $R_1$  does not preclude this behaviour. It can be proved as follows that this is not the case and the solution must cross the  $X$  axis in finite time and so leave  $R_1$ .

By considering an initial point  $(X_0, Y_0)$  in  $R_1$ , the following inequality holds

$$Y' = f(\tau) - \gamma Y - X^2 \leq -\gamma Y + (f_{max} - X_0^2). \quad (4)$$

Defining  $\bar{X}_0^2 = -(f_{max} - X_0^2)$ , which for a suitably large enough  $X_0$  is positive, and substituting this into (4) results in the inequality

$$Y' \leq -\gamma Y - \bar{X}_0^2. \quad (5)$$

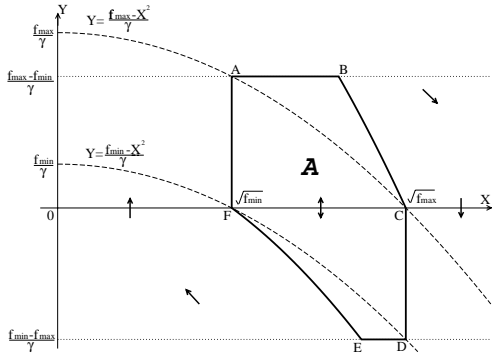


Figure 4: Absorbing set **A**.

Integrating (5) and solving for  $Y$  results in

$$\gamma Y \leq (\gamma Y_0 + \bar{X}_0^2) \exp(-\gamma \tau) - \bar{X}_0^2. \quad (6)$$

As  $\tau$  increases, (6) will tend to the inequality  $\gamma Y \leq -\bar{X}_0^2$ , which means  $Y$  must become negative in finite time, and so solutions must cross the  $X$  axis.

### 2.3. Example of the Flow of the System

Numerical solutions of (1) with the values  $\alpha = 1.069$ ,  $\beta = 0.428$ ,  $\gamma = 0.064$  and  $\mu = 2$  are shown in figure 3. Included in the diagrams are the  $Y_{max}$  and  $Y_{min}$  curves which are represented by dashed lines for the  $Y_{min}$  curve, and dotted lines for the  $Y_{max}$  curve. The top diagram shows the solution blowing up, the middle solution going to a period 1 limit cycle, and the bottom diagram the same period 1 limit cycle without the transient. Here the flow of (1) is seen to be what is expected from the above analysis.

### 2.4. Absorbing Set

An absorbing set **A** which contains all bounded periodic solutions of (1) is now shown. Additional constraints are that the inequality  $\gamma^2 \geq 8\sqrt{f_{max}}$  must hold, and that  $f(\tau)$  must be infinitely many times differentiable. When these constraints hold, the set **A** shown in figure 4 can be proved [8, 9] to be absorbing. The vertices of the set **A** are defined as

$$\begin{cases} A = (\sqrt{f_{min}}, (f_{max} - f_{min})/\gamma), \\ B = (\sqrt{f_{max} - (f_{max} - f_{min})/\lambda_u}, (f_{max} - f_{min})/\gamma), \\ C = (\sqrt{f_{max}}, 0), \\ D = (\sqrt{f_{max}}, (f_{min} - f_{max})/\gamma), \\ E = (\sqrt{f_{min} + (f_{max} - f_{min})/\lambda_l}, (f_{min} - f_{max})/\gamma), \\ F = (\sqrt{f_{min}}, 0), \end{cases}$$

and the edges are straight lines except BC, which is defined by the curve  $Y = \lambda_u(f_{max} - X^2)/\gamma$ , and EF, defined by the

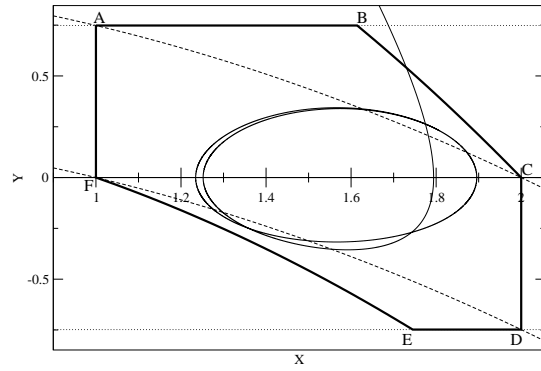


Figure 5: Period 1 solution of (1) when  $\alpha = 2.5$ ,  $\beta = 1.5$ ,  $\gamma = 4.01$  and  $\mu = 2$ , is contained within the absorbing set.

curve  $Y = \lambda_l(f_{min} - X^2)/\gamma$  where

$$\lambda_u = \frac{\gamma^2}{4\sqrt{f_{max}}} \left( 1 + \sqrt{1 - 8\sqrt{f_{max}}/\gamma^2} \right),$$

and  $\lambda_l$  is the smallest real root of

$$4f_{min}\lambda_l^4 + 4(f_{max} - f_{min})\lambda_l^3 - \gamma^4(\lambda_l - 1)^2 = 0.$$

It can be proved that if  $\gamma^2 \geq 8\sqrt{f_{max}}$ , then  $\lambda_l \in (1, 2]$ .

### 2.5. Example of the Absorbing Set

A numerical solution of (1) with the values  $\alpha = 2.5$ ,  $\beta = 1.5$ ,  $\gamma = 4.01$  and  $\mu = 2$  is shown in figure 5, with the absorbing set **A** included. It is seen clearly that the period 1 solution is contained within the absorbing set. It is conjectured that all periodic solutions satisfying the above constraints are period 1 solutions.

### 3. Coexisting Periodic Solutions

An exploration of the solutions of the dynamical system (1) with varying values of  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\mu$  demonstrates coexisting periodic solutions when the initial conditions are varied. Basin of attraction diagrams can be computed which identify these different periodic solutions. Shown in figure 6 are the coexisting periodic (periods 1, 3 and 5) solutions of (1) when  $\alpha = 3$ ,  $\beta = 2.5$ ,  $\gamma = 0.01$  and  $\mu = 2$ . Shown in figure 7 are the coexisting periodic (periods 1 and 2) solutions of (1) when  $\alpha = 3.5$ ,  $\beta = 2$ ,  $\gamma = 0.005$  and  $\mu = 1.67$ . Also found, but not shown here for  $\mu = 1.67$  and other  $\alpha$ ,  $\beta$  and  $\gamma$  values are period 3 and period 5 solutions. The parameter values  $\alpha = 6$ ,  $\beta = 5$ ,  $\gamma = 0.0017$  and  $\mu = 1.67$  result in period 1 and period 3 solutions, whilst the parameter values  $\alpha = 2$ ,  $\beta = 1.4$ ,  $\gamma = 0.0009$  and  $\mu = 1.67$  result in period 1 and period 5 solutions. The parameter  $\mu = 1.67$  was chosen from capacitance measurements on a practical diode (BB304). Circuit experiments which are currently being carried out in the lab will hopefully confirm the validity of these simulations.

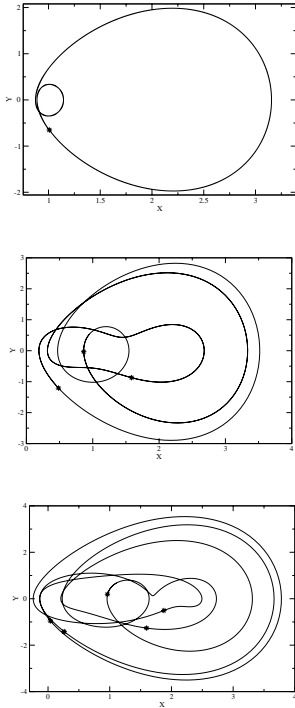


Figure 6: Co-existing periodic solutions of (1) when  $\alpha = 3$ ,  $\beta = 2.5$ ,  $\gamma = 0.01$  and  $\mu = 2$ . Top: Period 1 solution. Middle: Period 3 solution. Bottom: Period 5 solution. The stars denote the solution at  $t = 0, 2\pi, 4\pi, \dots$

#### 4. Conclusions

It has been shown in this paper that the driven series RL-diode circuit, where the diode operates only in the depletion region, possesses many interesting dynamical features, which can be examined both numerically and analytically. Phase plane analysis shows that a region of the phase plane can be defined for which the solution will always blow up. It also shows the direction of the flow of (1), and proves that the solution must cross the X axis. Also defined when certain constraints hold is an absorbing set which contains all bounded periodic solutions. Various subharmonic orders have also been found.

#### Acknowledgments

The authors would like to thank Michele Bartuccelli and Guido Gentile for many profitable discussions.

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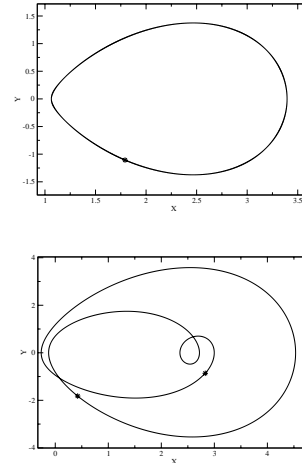


Figure 7: Co-existing periodic solutions of (1) when  $\alpha = 3.5$ ,  $\beta = 2$ ,  $\gamma = 0.005$ ,  $\mu = 1.67$ . Top: Period 1 solution. Bottom: Period 2 solution. The stars denote a  $2\pi$  time interval.

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