Synchronization of the Infinite-dimensional Chaotic Dynamical System

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Abstract The synchronization of chaotic dynamical systems has aroused much interest in recent years, both in theoretical and experimental studies. Among them, the synchronization of infinite-dimensional chaotic dynamical systems has special significance. In this paper, a method is proposed to realize the synchronization of the Mackey-Glass system. This is a delay differential equation and is infinite dimensional. We approximate the MG equation by a high-dimensional map, and then the synchronization of the map is studied. Numerical simulations have been done for the map and the main performance of the synchronization has been checked including the rate and precision of convergence. The potential applications of this synchronization method are also discussed finally.

Key words synchronization of chaotic systems, infinite-dimensional dynamical system, the Mackey-Glass equation

1. Introduction

Synchronization of chaotic systems has aroused much interest in recent years, particularly in light of potential applications of this phenomenon in secure communication, biology and life science, etc. Although different methods for constructing synchronized chaotic systems have been proposed [1][2][5], most of them concern low-dimensional systems with one positive Lyapunov exponent. It was turned out that information masked by such simple chaotic signals could be sometimes easily extracted [3], so it is desirable to use high-dimensional systems with multiple positive Lyapunov exponents (superchaotic systems). There are some methods for synchronizing superchaotic systems [4], but they have been applied only to finite-dimensional systems described by ordinary differential equations. Comparatively, synchronization of delay differential equations (DDE's) has special significance in that systems governed by DDE's have an infinite-dimensional state space and can produce superchaos with an arbitrarily large number of positive Lyapunov exponents in principle. A typical example of such infinite dimensional systems is the Mackey-Glass (MG) system.

The MG equation was first proposed by M. C. Mackey and L. Glass in 1977 to simulate the hematopoiesis [6]. The equation takes the follow form.

$$\dot{x} = a\frac{\theta^n x_\tau}{\theta^n + x_\tau^n} - bx \qquad (1)$$

where x(t) is the density of the mature circulating blood cells and $x_\tau = x(t-\tau)$; θ, n, a and b are positive constants usually fixed at $\theta = 1, n = 10, a = 0.2, b = 0.1$; b represents the decay of the blood cells while $a\frac{\theta^n x_\tau}{\theta^n + x_\tau^n}$ represents the production of the blood cells. Since it takes roughly four days for a new blood cell to mature, there is a time delay τ in the equation. With the increase of the delay time τ, the MG equation displays different dynamics from fixed point, limited cycle to chaos, which are qualitatively similar to the clinical observations [6].

The main reason that we choose the MG equation for our study is that it has been thoroughly investigated by many people [7] [8] and the properties of this equation are well known. For example, Farmer has shown that the number of positive Lyapunov exponents as well as dimension of strange attractor of this system can easily be controlled by varying the delay time τ. Furthermore, for fixed values of the parameters θ, n, a and b, dynamical characteristic quatities increase linearly with the increase of τ [7]. On the other hand, because the MG equation was first used as a theoretical model for physiological control as we mentioned before, we can expect the synchronization of it has some potential significance in medicine and physiology.

Some approaches to the synchronization of the MG equation have been proposed [9][10]. In this paper, we give our own.

2. Our synchronization scheme

In 1998, K. Pyragas [9] gave his method for synchronizing the MG equation and applied it to secure communication. He constructed a communication system with the equations of the sender, the transmitted signal, and the receiver given by

$$\dot{x} = \frac{a(x(t-\tau)+i(t))}{\theta^c + (x(t-\tau)+i(t))^c} - bx \qquad \text{sender} \qquad (2)$$

$$s(t) = x(t-\tau) + i(t) \qquad \text{transmitted signal} \qquad (3)$$

$$\dot{y} = \frac{as(t)}{\theta^c + s(t)^n} - cy \qquad \text{receiver} \qquad (4)$$

where $i(t)$ is the information signal. He verified that the variables x, y can be synchronized, that is $x=y$, when $\tau \to \infty$, so the information $i(t)$ can be recovered at the receiver.

$$i_R = s(t) - y(t-\tau) \qquad (5)$$

We know that in 1990, Pecora and Carroll [1] showed that when a state variable from a chaotically evolving system was transmitted as an input to a replica of part of the original system, the replica subsystem (response system) could synchronize to the original system (drive system). This method is usually called the drive- response model. Seeing the method of Pyragas from this view, we find that it is just using $x(t-\tau)$, which is visible in the original equation, as the drive signal. But don't forget that the MG equation is a delay differential equation and it is infinite-dimensional. We feel by intuition that $x(t-\tau)$ is not the only one that we can use as the drive signal though this is not obvious in the original equation. We can approximate the MG differential equation by a high-dimensional map, and in this form, we have many signals can be used. This method was not invented by us. In fact, Farmer used it to compute the Lyapunov exponents of the MG equation [7].

According to above considerations, we get our own synchronization scheme.

Firstly, we approximate the MG equation by a high-dimensional map. Choosing any integration scheme, for example, Euler integration, the MG equation can be approximated by a map

$$x(t+\Delta t) = x(t) + F(x, x_\tau)\Delta t \qquad (6)$$

where $F(x, x_\tau)$ is the MG equation, that is

$$F(x, x_\tau) = a\frac{\theta^n x_\tau}{\theta^n + x_\tau^n} - bx$$

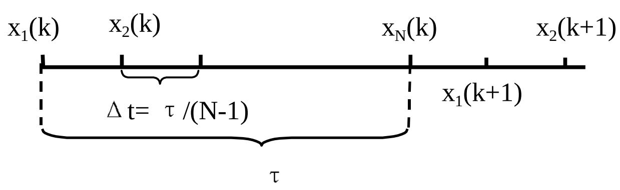
And equation (6) can be written into an N-dimensional iterated map.

$$\begin{cases} x_1(k+1) = x_N(k) + F(x_N(k), x_1(k))\Delta t \\ x_2(k+1) = x_1(k+1) + F(x_1(k+1), x_2(k))\Delta t \\ \qquad \vdots \\ x_N(k+1) = x_{N-1}(k+1) + F(x_{N-1}(k+1), x_N(k))\Delta t \end{cases} \qquad (7)$$

where $x_i (i=1,2,\cdots,N)$ denote the samples taken at intervals $\Delta t = \tau/(N-1)$, i.e.

$$(x_1, \cdots, x_{N-1}, x_N) = (x(t-(N-1)\Delta t), \cdots, x(t-\Delta t), x(t))$$

k labels the iteration and each iteration will move the system forward by time $\tau + \Delta t$. This can be illustrated by the follow figure.



Now we have approximated the original MG equation (1) by an N-dimensional map (7).

Secondly, we try to synchronize equation (7) using the drive-response model. The drive system is governed by equation (7) while the response system is governed by the follow equations.

$$\begin{cases} x_1'(k+1) = x_N'(k) + F(x_N'(k), x_1'(k))\Delta t \\ x_2'(k+1) = x_1'(k+1) + F(x_1'(k+1), x_2'(k))\Delta t \\ \vdots \\ x_i'(k+1) = x_i(k+1) \\ x_{i+1}(k+1) = x_i(k+1) + F(x_i(k+1), x_{i+1}(k))\Delta t \\ \vdots \\ x_N(k+1) = x_{N-1}(k+1) + F(x_{N-1}(k+1), x_N(k))\Delta t \end{cases} \quad (8)$$

Now we can arbitrarily choose x_i from the N signals $(x_1, x_2 \cdots, x_{N-1})$ as the drive signal and $x(t-\tau)$ is no longer the only selection.

Numerical simulations have shown that the other signals of the response system can be synchronized with the corresponding signals of the drive system and $x_i(i=1,2,\cdots,N)$ remain the qualitative features of the original signal x.

3. Synchronization results

Numerical simulations have been done for different delay parameters τ and the typical results are as follows

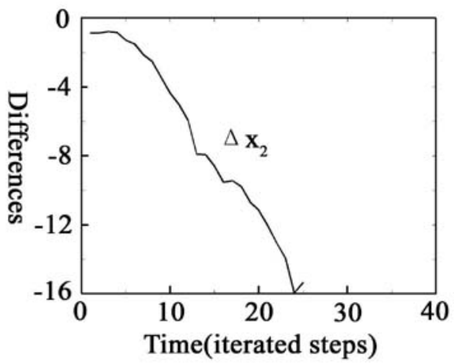


FIG. 1. $\tau=17, \Delta t = 0.1, N=\tau/\Delta t+1=171$, i.e. there are 171 signals in total,（$x_1 \cdots x_{171}$）. Arbitrarily selecting x_i (x_{30} is selected here) as the drive signal, then the other 170 signals of the response system have all been synchronized with the corresponding signals of the drive system. Here the result of x_2 is given. The figure shows the log of the absolute value of the difference between the signals of the drive and response systems. The precision of the synchronization is very high with the remain difference <10^{-16}, and the synchronization can be achieved within 30 iterated steps.

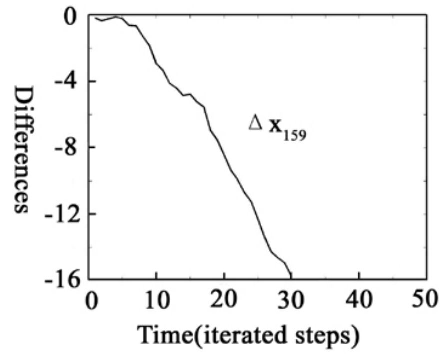


FIG. 2. $\tau=30, \Delta t=0.1, N=\tau/\Delta t+1=301$, i.e. there are 301 signals in total,（$x_1 \cdots x_{301}$）. Here x_4 is selected as the drive signal, then the other 300 signals of the response system have all been synchronized with the corresponding signals of the drive system. The result of x_{159} is given here.

4. Power spectra of the signals

As we have mentioned, $x_i(i=1,2,...,N)$ in equation (7) is actually sampled from the original continuous signal x at intervals $\Delta t=\tau/(N-1)$. Our numerical simulation shows that these sampled signals still remain the qualitative features of the original signal. The power spectra and the phase plots of x and x_i (i was arbitrarily selected) are almost the same.

Here the power spectra of x and x_i are given.

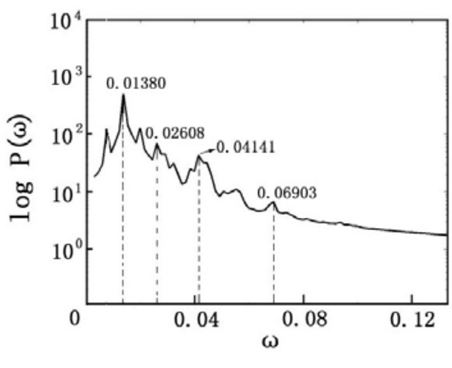


FIG. 3. the powerspectra of the signal x created by the MG equation (1) when $\tau=17$.

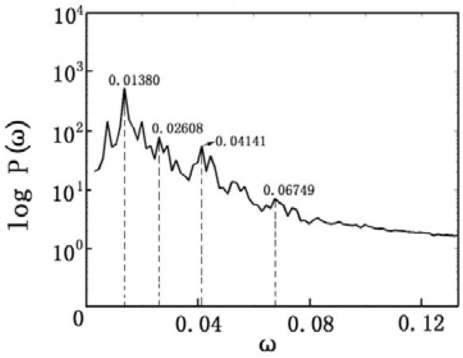


FIG. 4. The power spectra of x_1 created by the approximated map (7) which is very similar to FIG. 3. $\tau = 17, \Delta t = 0.1, N = \tau/\Delta t + 1 = 171$. The power spectra of other signals $x_i\,(i = 1,2,...171 \quad i \neq 1)$ are almost the same.

These results shows that a single approximated signal still carries the information of the original whole signal and can in some ways explain why that only one signal can lead to the synchronization of many other signals.

5. Discussion

In this paper, a method is given to synchronize the MG equation which is a delay differential equation and is infinite dimensional. We firstly approximate the MG equation by a finite dimensional map, and then synchronize the map using the drive-response model. Numerical simulation results show that the rate and precision of the synchronization is pretty good. Furthermore, the signals of the high-dimensional map still remain the qualitative features of the original signal of the differential equation (1).

This method is efficient in that one drive signal can lead to the synchronization of many other signals. This phenomenon can easily make us think of the potential application of it in communication. On the other hand, we find the form of the high-dimensional map is somewhat similar to the spatiotemporal chaos which indicate there are some common features between these two phenomena.

Of course, all these potential significances have to be verified which will be done in our further work.

References

[1] L. M. Pecora and T. L. Carrol (1991) *"Driving systems with chaotic signals."* Physical Review A, 44: 2374-2383

[2] L. Kocarev, K. S. Halle, K. Eckert, U. Parlitz and L. O. Chua (1992) *"Experimental demonstration of secure communication via chaotic synchronization."* International Journal of Bifurcation and chaos, 2(3) : 709-713

[3] G. Perez and H. A. Cerdeira (1995) *"Extrating messages masked by chaos."* Physical Review Letters, 74: 1970-1973

[4] J. H. Peng, E. J. Ding, M. Ding and W. Yang (1996) *"Synchronization* hyperchaos *with a scalar transmitted signal."* Physical Review Letters, 76: 904-907

[5] L. Kocarev and U. Parlitz (1995) *"General approach for chaotic synchronization with applications to communication."* Physical Review Letters, 74(25): 5028-5031

[6] M. C. Mackey and L. Glass (1977) *"Oscillations and chaos in physiological control systems."* Science, 197: 287-289

[7] J. D. Farmer (1982) *"Chaotic attractors of infinite-dimensional dynamical systems."* physica D, 4: 366-393

[8] A. Namajunas, K. Pyragas and A. Tamasevicius (1995) *"An electronic analog of the Mackey-Glass system."* Physics Letters A, 201: 42 – 46

[9] K. Pyragas (1998) *"Transmission of signal via synchronization of chaotic time-delay systems."* International Journal of Bifurcation and chaos, 2 (9) : 1839-1842

[10] M. J. *Bünner* and Wolfram Just (1998) *"Synchronization of time-delay systems."* Physics Letters E, 58(4): 4072 – 4075