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Recent Developments in Quantum Statistics

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Joint Work with K. Yamagata and R. D. Gill

Plan of talk

Motivation

- Part I: Review of quantum estimation theory
- Part II: Review of classical local asymptotic normality
- Part III: Local asymptotic normality in the quantum domain

Part I: Review of quantum estimation theory

Classical estimation

- Statistical model:

$$\{P_\theta\}_{\theta\in\Theta\subset\mathbb{R}^d} \text{ on } (\Omega, \mathcal{F}, \mu) \quad \xrightarrow{P_\theta \ll \mu} \quad \left\{p_\theta := \frac{dP_\theta}{d\mu}\right\}_{\theta\in\Theta}$$

- Estimator: $\hat{\theta} : \Omega \longrightarrow \Theta$
- Unbiased estimator:

$$E_{P_\theta}[\hat{\theta}] := \int_\Omega \hat{\theta}(\omega) p_\theta(\omega) \mu(d\omega) = \theta \qquad (\forall \theta \in \Theta)$$

- Fisher information matrix $J_\theta = [(J_\theta)_{ij}]$

$$(J_\theta)_{ij} := E_{P_\theta}\left[\boxed{\partial_i \log p_\theta}\,(\partial_j \log p_\theta)\right]$$

$$(\partial_i := \partial/\partial\theta^i)$$

Logarithmic Derivative

- Cramér-Rao inequality:

$$V_{P_\theta}[\hat{\theta}] \geq (J_\theta)^{-1}$$

for any unbiased estimator $\hat{\theta}$

Quantum estimation

1) Quantum statistical model

$$\mathcal{S} = \{\rho_\theta (> 0) : \ \theta = (\theta^1, \dots, \theta^d) \in \Theta \subset \mathbb{R}^d\}$$

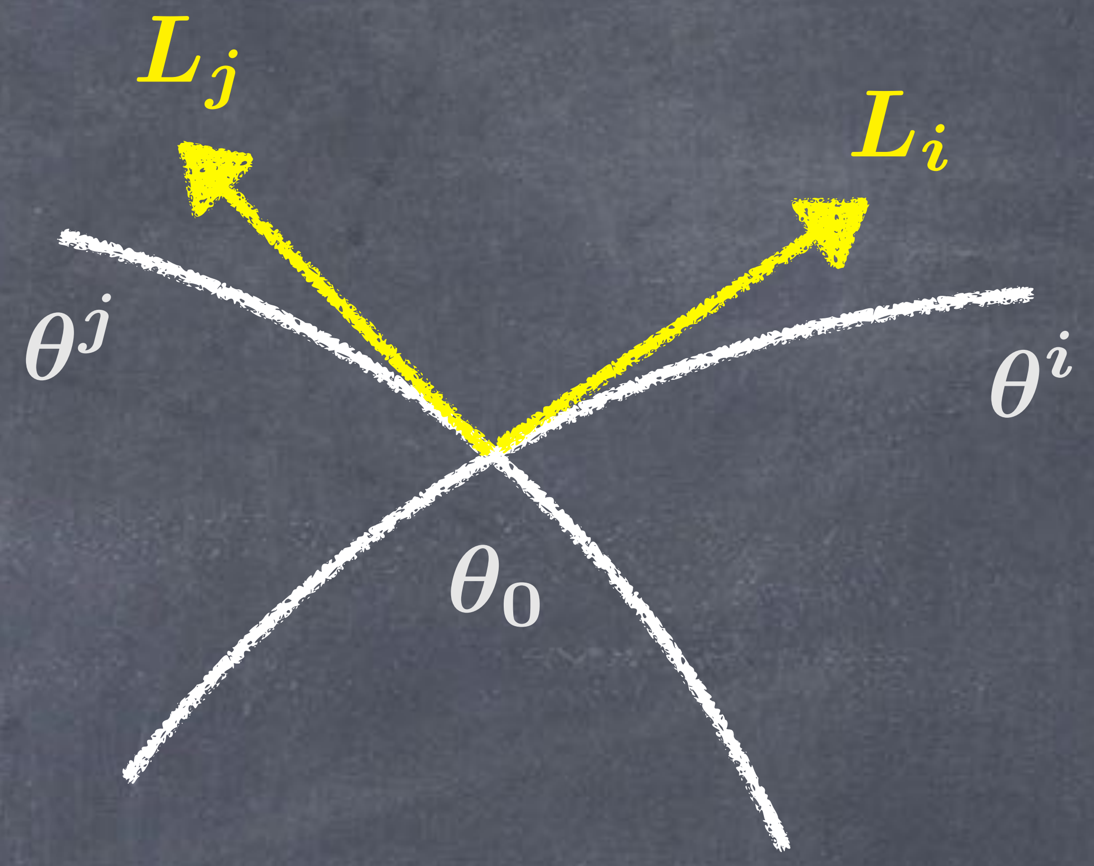
2) Symmetric Logarithmic Derivative (SLD):

$$\partial_i \rho_\theta = \frac{1}{2}(\rho_\theta L_i + L_i \rho_\theta)$$

3) SLD Fisher information matrix $J_\theta^S = [(J_\theta^S)_{ij}]$

$$(J_\theta^S)_{ij} = \mathrm{Re}(\mathrm{Tr}\, \rho_\theta L_j L_i)$$

4) SLD Cramér-Rao inequality:



$$V_{\rho_\theta}[M] \geq (J^S_\theta)^{-1}$$

with equality iff SLDs commute
and M the simultaneous spectral measure

5) It is thus customary to switch the target to

$$\min_M \ \mathrm{tr}\, W V_{\rho_\theta}[M]$$

given a weight matrix $W > 0$

6) Holevo bound

$$\operatorname{tr} W V_{\rho_{\theta_0}}[M] \geq C^{H}_{\theta_0}(\rho_\theta, W)$$

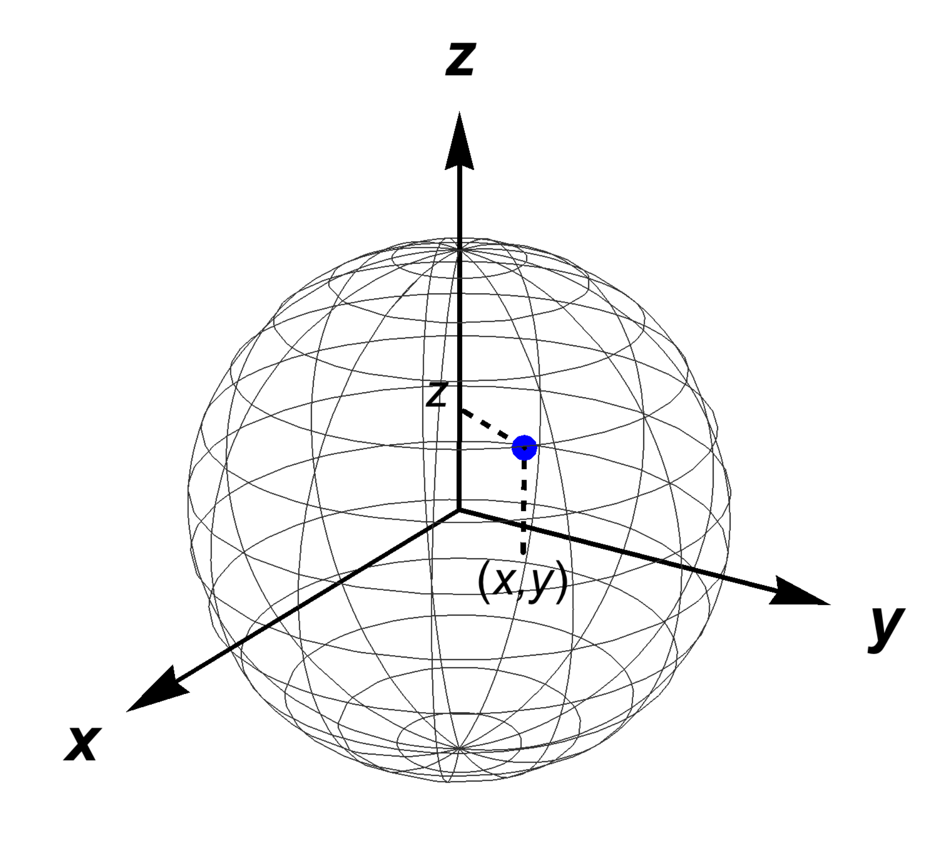
where

$$C^{\mathrm{H}}_{\theta_0}(\rho_\theta, W) := \min_{V,B}\{\operatorname{tr} W V \ : \ V \geq [\mathrm{Tr}\rho_{\theta_0} B_j B_i],$$
$$\operatorname{Re}\operatorname{Tr}\rho_{\theta_0} L_j B_i = \delta_{ij}\}$$

with V a real symmetric matrix and

$B_1, \ldots, B_d$ selfadjoint operators

Eg. qubit estimation



Qubit state model:

$$\mathcal{S} = \left\{ \rho_{(x,y,z)} = \frac{1}{2}(I + x\sigma_x + y\sigma_y + z\sigma_z) \,:\, x^2 + y^2 + z^2 < 1 \right\}$$

Hayashi-Gill-Massar bound:

$$\min_M \mathrm{tr} W\, V_\theta(M) = C_\theta^{\mathrm{HGM}}(\rho_\theta, W) := \left(\mathrm{tr}\sqrt{(J_\theta^S)^{-1/2}\, W\, (J_\theta^S)^{-1/2}} \right)^2$$

When $W := J_\theta^S$

$$C_\theta^{\mathrm{HGM}}(\rho_\theta, J_\theta^S) = 9 \quad \text{while} \quad C_\theta^{\mathrm{H}}(\rho_\theta, J_\theta^S) = 3 + 2\|\theta\|$$

Motivation

separable

$$\begin{array}{ccccccc} \rho_\theta & \otimes & \rho_\theta & \otimes & \cdots & \otimes & \rho_\theta \\ \uparrow & & \uparrow & & & & \uparrow \\ M_1 & \otimes & M_2 & \otimes & \cdots & \otimes & M_n \end{array}$$

collective

$$\underbrace{\rho_\theta \otimes \rho_\theta \otimes \cdots \otimes \rho_\theta}_{M^{(n)}}$$

- Suppose we have n copies of a quantum system each in the same state
- We are allowed to use "collective" measurement
- What is the best we can do as $n \to \infty$?
- We study this problem by extending the theory of LAN to the quantum domain

Main result (i.i.d. case)

- [Direct part]
  The Holevo bound is achievable.

- [Converse part]
  The Holevo bound is unbreakable.

Main result (generic case)

- [Direct part]
  The <u>asymptotic representation bound</u> is achievable.
- [Converse part]
  The <u>asymptotic representation bound</u> is unbreakable.

Part II: Review of classical local asymptotic normality

Local Asymptotic Normality

A sequence of models $\{P_\theta^{(n)} : \theta \in \Theta \subset \mathbb{R}^d\}$

is called LAN at $\theta_0 \in \Theta$ if

$$\log \frac{dP^{(n)}_{\theta_0 + h/\sqrt{n}}}{dP^{(n)}_{\theta_0}} = h^i \Delta_i^{(n)} - \frac{1}{2} h^i h^j J_{ij} + o_{P_{\theta_0}}(1)$$

where

$$\Delta^{(n)} \overset{P^{(n)}_{\theta_0}}{\rightsquigarrow} N(0, J)$$

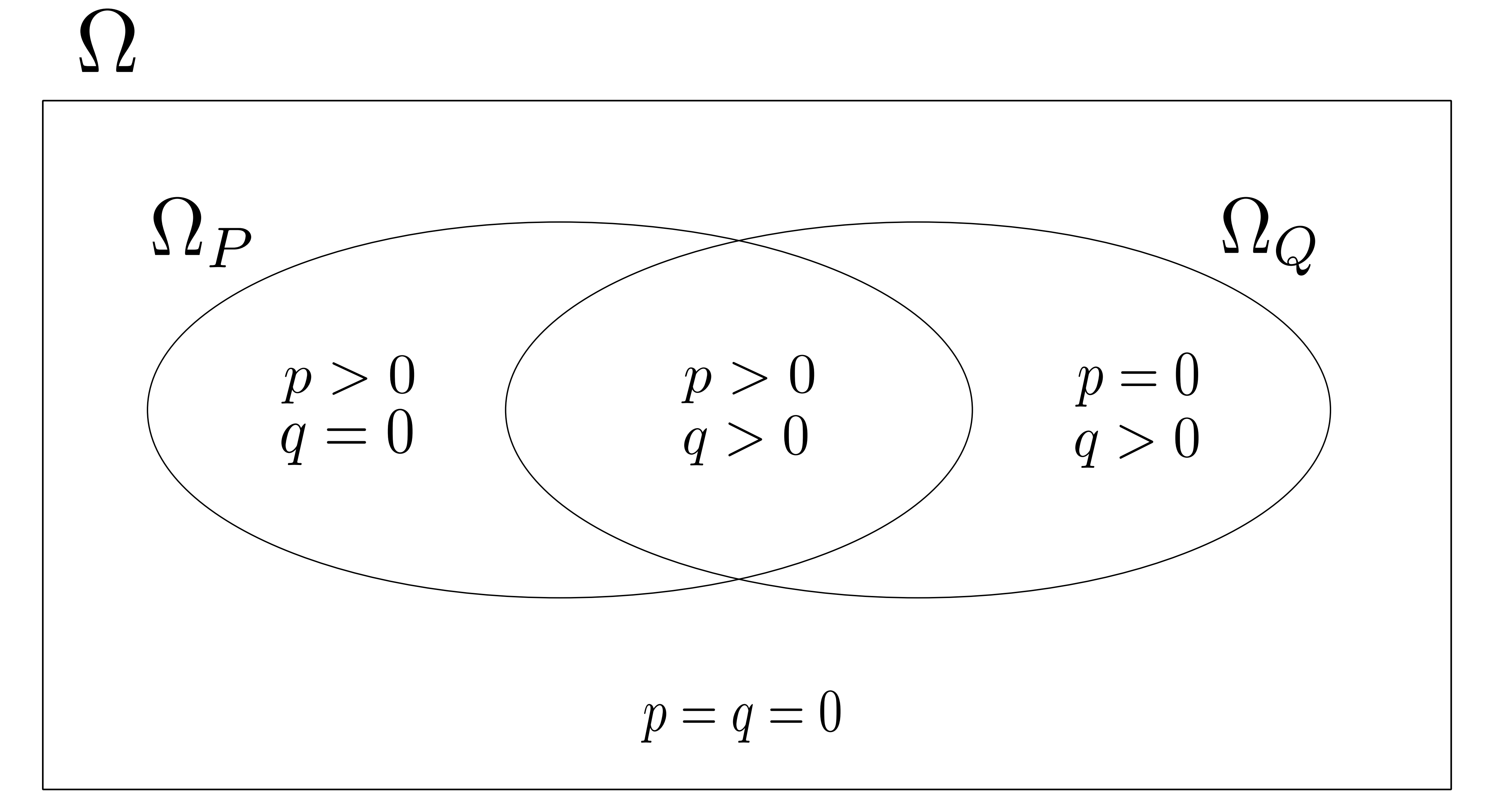
NB. Given P and Q, let

$$Q = Q^{ac} + Q^{\perp} \qquad (Q^{ac} \ll P,\ Q^{\perp} \perp P)$$

be the Lebesgue decomposition. Then

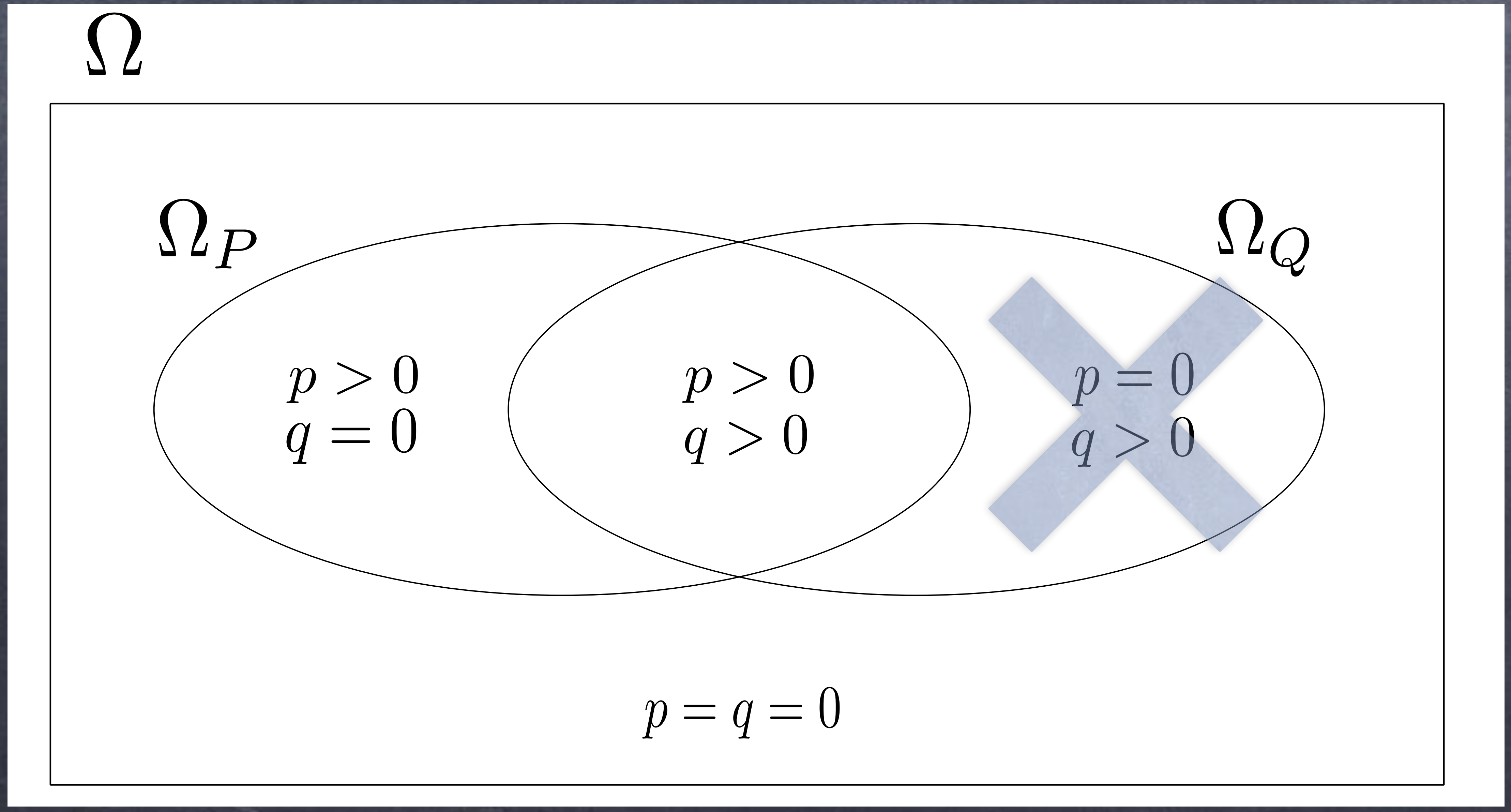
$$\frac{dQ}{dP}(\omega) := \frac{dQ^{ac}}{dP}(\omega)$$

$$= \frac{q^{ac}(\omega)}{p(\omega)}$$



Radon-Nikodym theorem

$$Q \ll P \implies dQ = \frac{dQ}{dP} dP$$



Local Asymptotic Normality

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where

$$\boldsymbol{\Delta}^{(n)} \overset{P^{(n)}_{\theta_0}}{\rightsquigarrow} N(\mathbf{0}, J)$$

Prototype of LAN: $p_\theta^{(n)} = p_\theta^{\otimes n}$

$$\log \frac{p_{\theta_0+h/\sqrt{n}}^{\otimes n}}{p_{\theta_0}^{\otimes n}}(X_1, \ldots, X_n)$$

$$= h^i \underbrace{\left\{ \frac{1}{\sqrt{n}} \sum_{k=1}^n \partial_i \log p_{\theta_0}(X_k) \right\}}_{\Delta_i^{(n)}(X_1,\ldots,X_n)} - \frac{1}{2} h^i h^j \underbrace{\left\{ -\frac{1}{n} \sum_{k=1}^n \partial_i \partial_j \log p_{\theta_0}(X_k) \right\}}_{J_{ij} + o_{p_{\theta_0}}(1)} + o(\frac{1}{n})$$

where

$$\Delta^{(n)} \overset{p_{\theta_0}^{\otimes n}}{\rightsquigarrow} N(0, J)$$

with J being the Fisher information matrix

Similarity to Gaussian shift model

LAN

$$\log \frac{dP^{(n)}_{\theta_0+h/\sqrt{n}}}{dP^{(n)}_{\theta_0}} = h^i \Delta_i^{(n)} - \frac{1}{2} h^i h^j J_{ij} + o_{p_{\theta_0}}(1)$$

Gaussian shift model

$$\log \frac{dN(Jh, J)}{dN(0, J)}(X_1, \ldots, X_d) = h^i X_j - \frac{1}{2} h^i h^j J_{ij}$$

Contiguity

A sequence $Q^{(n)}$ of probability measures is called <u>contiguous</u> to another sequence $P^{(n)}$ of probability measures, denoted $Q^{(n)} \lhd P^{(n)}$, if

$$P^{(n)}(A^{(n)}) \to 0 \implies Q^{(n)}(A^{(n)}) \to 0$$

contiguous = asymptotically absolutely continuous

Le Cam's third lemma

Theorem (Radon-Nikodym)

$$Q \ll P \implies dQ = \frac{dQ}{dP}dP$$

weak convergence analogue

Theorem (Le Cam)

If $Q^{(n)} \lhd P^{(n)}$ and $\left(X^{(n)}, \frac{dQ^{(n)}}{dP^{(n)}}\right) \overset{P^{(n)}}{\rightsquigarrow} (X, V)$, then

$$X^{(n)} \overset{Q^{(n)}}{\rightsquigarrow} \mathcal{L}(\bullet) := E[\mathbb{1}_{\bullet}(X)V]$$

Le Cam's third lemma

(Gaussian version)

Theorem

If

$$\begin{pmatrix} X^{(n)} \\ \log \frac{dQ^{(n)}}{dP^{(n)}} \end{pmatrix} \overset{P^{(n)}}{\rightsquigarrow} N\left(\begin{pmatrix} 0 \\ -\frac{1}{2}\sigma^2 \end{pmatrix}, \begin{pmatrix} \Sigma & \tau \\ \tau^\top & \sigma^2 \end{pmatrix} \right)$$

then ($Q^{(n)} \triangleleft P^{(n)}$ and)

$$X^{(n)} \overset{Q^{(n)}}{\rightsquigarrow} N(\tau, \Sigma)$$

Third lemma under LAN

Suppose

$$\log \frac{dP^{(n)}_{\theta_0+h/\sqrt{n}}}{dP^{(n)}_{\theta_0}} = h^i \Delta^{(n)}_i - \frac{1}{2} h^i h^j J_{ij} + o_{p_{\theta_0}}(1)$$

and

$$\begin{pmatrix} X^{(n)} \\ \Delta^{(n)} \end{pmatrix} \overset{P^{(n)}_{\theta_0}}{\rightsquigarrow} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma & \tau \\ \tau^\top & J \end{pmatrix} \right)$$

Then

$$X^{(n)} \overset{P^{(n)}_{\theta_0+h/\sqrt{n}}}{\rightsquigarrow} N(\tau h, \Sigma)$$

The moral:
LAN model is locally asymptotically similar to Gaussian shift model

Asymptotic Representation Theorem

Assume that

1) $\{P_\theta^{(n)} : \theta \in \Theta \subset \mathbb{R}^d\}$ is LAN at $\theta_0 \in \Theta$

2) Seq. of statistics $T^{(n)}$ on $P^{(n)}_{\theta_0 + h/\sqrt{n}}$ is weakly convergent for each h (i.e. $T^{(n)} \overset{h}{\rightsquigarrow} \exists \mathcal{L}_h$)

Then there exists a statistic T on $N(Jh, J)$ s.t.

$$T^{(n)} \overset{h}{\rightsquigarrow} T$$

$T = T(\Delta, Z)$ $(\Delta \sim N(Jh, J))$

for all h

Part III: LAN in the quantum domain

【 Annals of Statistics, 41, 2197–2217 (2013) 】

【 Bernoulli, 26, 2105–2142 (2020) 】

【 Annals of Statistics, 51, 1159–1182 (2023) 】

Difficulties in extending LAN to the quantum domain

LAN: $$\log \frac{dP^{(n)}_{\theta_0+h/\sqrt{n}}}{dP^{(n)}_{\theta_0}} = h^i\Delta^{(n)}_i - \frac{1}{2}h^i h^j J_{ij} + o_{P_{\theta_0}}(1)$$

$$\left(\Delta^{(n)} \overset{\theta_0}{\rightsquigarrow} N(0,J)\right)$$

What is the quantum counterpart of

1) Radon-Nikodym derivative?
2) contiguity and third lemma?
3) weak convergence?

History of quantum LAN

- Guta and Kahn's strong q-LAN (2006, 2009)

$$\lim_{n\to\infty} \sup_{h\in K^{(n)}} \left\| \sigma_h - \Gamma^{(n)}(\rho^{\otimes n}_{\theta_0+h/\sqrt{n}}) \right\|_1 = 0$$

and

$$\lim_{n\to\infty} \sup_{h\in K^{(n)}} \left\| \Lambda^{(n)}(\sigma_h) - \rho^{\otimes n}_{\theta_0+h/\sqrt{n}} \right\|_1 = 0$$

Drawbacks

- iid
- nondegeracy

- Guta and Jencova's weak q-LAN (2007)

Radon-Nikodym derivative in the quantum domain?

f-divergence

- Classical

$$D_f(P\|Q) = \int_\Omega f\left(\frac{dQ}{dP}\right) dP$$

- Quantum

$$D_f(\rho\|\sigma) = \mathrm{Tr}\,\rho\left\{f\left(\frac{\mathbf{L}_\sigma}{\mathbf{R}_\rho}\right) I\right\}$$

Sufficiency

- $\mathcal{G}$ ($\subset \mathcal{F}$) is sufficient for $\{P, Q\}$

$\forall A \in \mathcal{F}, \exists h \in m\mathcal{G}, \forall \mu \in \{P, Q\}$
$E_\mu[\mathbb{1}_A|\mathcal{G}] = h \quad (\mu\text{-a.e.})$

$$\iff \quad D(P\|Q) = D(P|_\mathcal{G}\|Q|_\mathcal{G})$$

$$\iff \quad \frac{dP}{dQ} \in m\mathcal{G}$$

- $\mathcal{A}$ ($\subset B(\mathcal{H})$) is sufficient for $\{\rho, \sigma\}$

$\exists \Gamma : \mathcal{A} \to B(\mathcal{H})$: CPTP
$\forall \tau \in \{\rho, \sigma\}, \ \Gamma(\tau|_\mathcal{A}) = \tau$

$$\iff \quad D(\rho\|\sigma) = D(\rho|_\mathcal{A}\|\sigma|_\mathcal{A})$$

$$\iff \quad \underbrace{\rho^{\sqrt{-1}t}\sigma^{-\sqrt{-1}t}}_{[D\rho, D\sigma]_t} \in \mathcal{A} \qquad (\forall t \in \mathbb{R})$$

Simple hypothesis testing

[Hiai-Petz (1991), Ogawa-Nagaoka (2000)]

- Classical likelihood ratio test (p vs. q)

$$\frac{1}{n}\log\frac{q^{\otimes n}}{p^{\otimes n}} > \lambda$$

- Quantum likelihood ratio test (ρ vs. σ)

$$\sigma^{\otimes n} - e^{n\lambda}\rho^{\otimes n} > 0$$

What about quantum parameter estimation?

- One may conceive a q-likelihood ratio via

$$\sigma - \lambda\rho = \sqrt{\rho}\left(\boxed{\sqrt{\rho^{-1}}\sigma\sqrt{\rho^{-1}}} - \lambda I\right)\sqrt{\rho}$$

- Unfortunately, this quantity is not useful in quantum estimation.
- A completely new idea is needed.

A breakthrough: quantum information geometry

$$\boxed{\rho_\theta = e^{\frac{1}{2}[\theta F - \psi(\theta)]}\,\rho_0\, e^{\frac{1}{2}[\theta F - \psi(\theta)]}}$$

SLD geodesic = quantum exp. family

Nice properties:

1) quantum Cramér-Rao lower bound is uniformly achievable
2) can be extended to non-faithful states

Absolute continuity & singularity

1) σ is absolutely continuous to ρ, denoted $\sigma \ll \rho$, if there is a positive operator $R\,(\geq 0)$ that satisfies $\boxed{\sigma = R\rho R}$

2) σ is singular to ρ, denoted $\sigma \perp \rho$, if

$$\operatorname{supp}\sigma \perp \operatorname{supp}\rho$$

An explicit expression

- For $\rho, \sigma > 0$

operator geometric mean

$$\sigma = R\rho R \iff R = \sigma \# \rho^{-1} = \sqrt{\sigma}\left(\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}}\right)^{-1}\sqrt{\sigma}$$

- For a generic case, the inverse is replaced by the generalized inverse.

Remark

- Classical view for absolute continuity:

$$\operatorname{supp}\sigma \subset \operatorname{supp}\rho$$

eg) $R = \sqrt{\rho^{-1}}\sigma\sqrt{\rho^{-1}} \iff \sigma = \sqrt{\rho}\,R\sqrt{\rho}$

- For pure states $\rho = |\xi\rangle\langle\xi|$ and $\sigma = |\eta\rangle\langle\eta|$

1) $\operatorname{supp}\sigma \subset \operatorname{supp}\rho \iff \sigma = \rho$

2) $\sigma \ll \rho \iff \langle\xi|\eta\rangle \neq 0$

$\sigma = R\rho R$

Quantum Lebesgue decomposition

<u>Theorem.</u> Given quantum states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, the decomposition

$$\sigma = \underbrace{R\rho R}_{\sigma^{ac}} + \underbrace{\tau}_{\sigma^{\perp}} \qquad (R \geq 0,\ \tau \geq 0,\ \tau \perp \rho)$$

uniquely exists.

In fact, let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ with

$$\mathcal{H}_1 := \ker\left(\sigma\rfloor_{\operatorname{supp}\rho}\right),\ \mathcal{H}_2 := \operatorname{supp}\left(\sigma\rfloor_{\operatorname{supp}\rho}\right),\ \mathcal{H}_3 := \ker\ \rho$$

$$\left(\sigma\rfloor_{\operatorname{supp}\rho} := \iota_\rho^*\,\sigma\,\iota_\rho\right)$$

and let

> 0

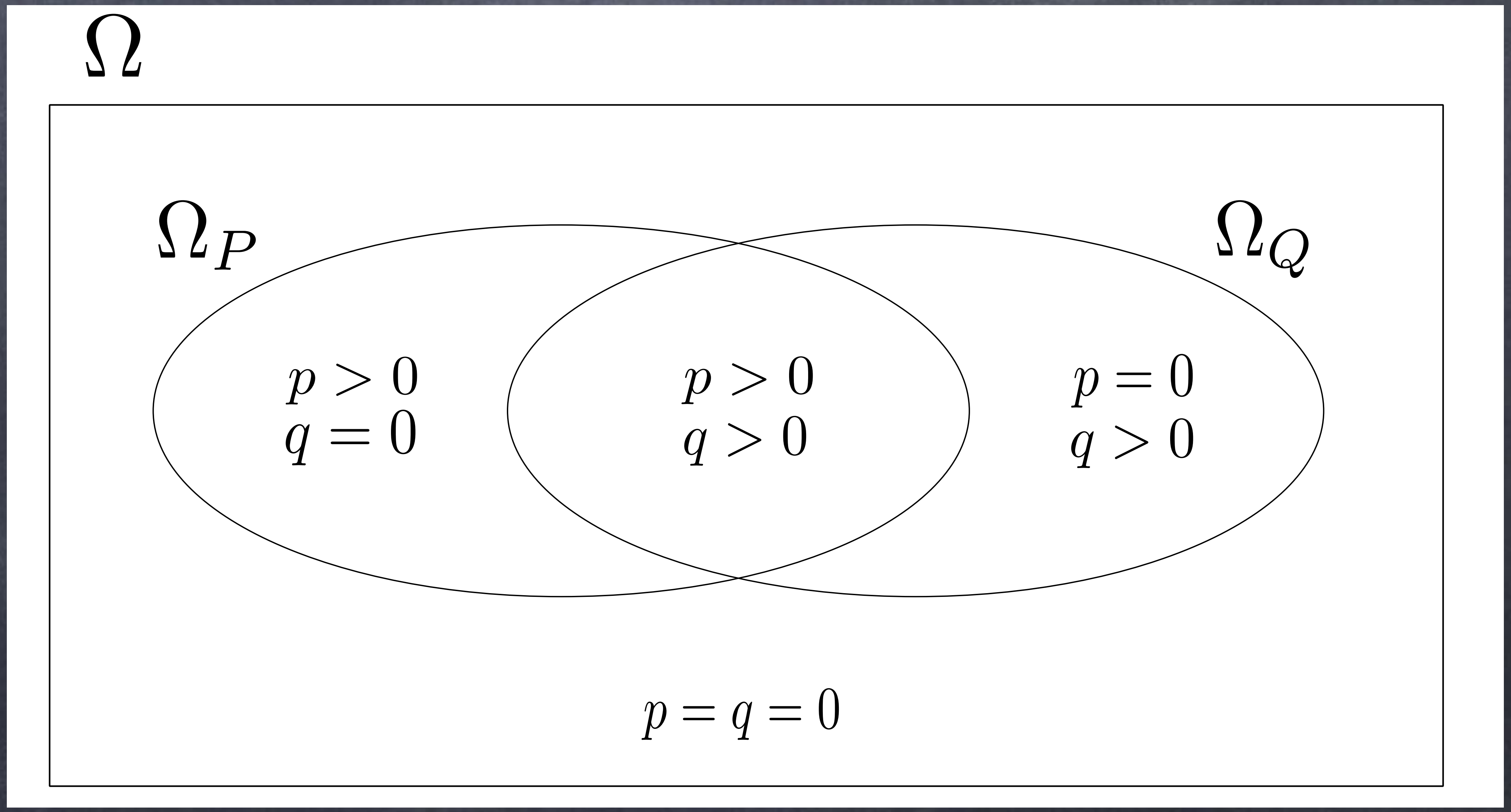
$$\rho = \begin{pmatrix} \rho_2 & \rho_1 & 0 \\ \rho_1^* & \rho_0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_0 & \alpha \\ 0 & \alpha^* & \beta \end{pmatrix}$$

$\sigma\rfloor_{\operatorname{supp}\rho}$

Then

$$\sigma^{ac} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_0 & \alpha \\ 0 & \alpha^* & \alpha^*\sigma_0^{-1}\alpha \end{pmatrix}$$

$$\sigma^{\perp} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta - \alpha^*\sigma_0^{-1}\alpha \end{pmatrix}$$



Remark

In general

$$\mathcal{H}_1 \neq \operatorname{supp}\rho \cap \ker\,\sigma \quad \text{and} \quad \mathcal{H}_2 \neq \operatorname{supp}\rho \cap \operatorname{supp}\sigma$$

In fact, for $\rho = |\xi\rangle\langle\xi|$ and $\sigma = |\eta\rangle\langle\eta|$

1) $\operatorname{supp}\rho \cap \operatorname{supp}\sigma \neq \{0\} \iff \rho = \sigma$

2) $\mathcal{H}_2 \neq \{0\} \iff \langle\xi|\eta\rangle \neq 0$

Quantum likelihood ratio

We call the positive operator R that satisfies

$$\sigma = R\rho R + \tau \qquad (R \geq 0,\ \tau \geq 0,\ \tau \perp \rho)$$

the <u>square-root likelihood ratio</u>, denoting $\mathcal{R}(\sigma|\rho)$

This is our solution to the problem of finding the noncommutative Radon-Nikodym derivative pertinent to quantum estimation theory. In fact, it allows us to extend the notions of contiguity, weak convergence, etc., to the quantum domain, establishing a theory of quantum local asymptotic normality.

Quantum LAN

$\mathcal{S}^{(n)} = \{\rho_\theta^{(n)} : \theta \in \Theta \subset \mathbb{R}^d\}$ is called <u>q-LAN</u> at $\theta_0 \in \Theta$ if

$R_h^{(n)} := \mathcal{R}\left(\rho^{(n)}_{\theta_0 + h/\sqrt{n}} \middle| \rho^{(n)}_{\theta_0}\right)$ is expanded in h as

$$\log\left(R_h^{(n)} + o_{L^2}(\rho^{(n)}_{\theta_0})\right)^2 = h^i \Delta_i^{(n)} - \frac{1}{2}\left(J_{ij} h^i h^j\right) I^{(n)} + o_D(h^i \Delta_i^{(n)}, \rho^{(n)}_{\theta_0})$$

where $\quad \Delta^{(n)} \overset{\rho^{(n)}_{\theta_0}}{\rightsquigarrow} N(0, J)$

Third lemma under q-LAN

Suppose $\mathcal{S}^{(n)} = \{\rho_\theta^{(n)} : \theta \in \Theta \subset \mathbb{R}^d\}$ is q-LAN at $\theta_0 \in \Theta$ and

$$\begin{pmatrix} X^{(n)} \\ \Delta^{(n)} \end{pmatrix} \overset{\rho_{\theta_0}^{(n)}}{\rightsquigarrow} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma & \tau \\ \tau^* & J \end{pmatrix}\right)$$

Then ($\rho_{\theta_0+h/\sqrt{n}}^{(n)} \triangleleft \rho_{\theta_0}^{(n)}$ and)

$$X^{(n)} \overset{\rho_{\theta_0+h/\sqrt{n}}^{(n)}}{\rightsquigarrow} N((\mathrm{Re}\,\tau)h, \Sigma)$$

The moral:
q-LAN model is locally asymptotically similar to q-Gaussian shift model

Embedding $\rho_\theta^{\otimes n}$ into Gaussian shift models

$$
\begin{tikzcd}
 & \text{q-LAN \& 3rd lemma} \arrow[dl, "X^{(n)}"'] \arrow[d, "D^{(n)}"] \arrow[dr, "Y^{(n)}"] & \\
N((\mathrm{Re}\tau_X)h, \Sigma_X) & N((\mathrm{Re}\tau_D)h, \Sigma_D) & N((\mathrm{Re}\tau_Y)h, \Sigma_Y)
\end{tikzcd}
$$

Choose $D^{(n)}$ suitably so that the Holevo bound of the resulting q-Gaussian is identical to that of ρ_θ

Asymptotic q-Repr. Theorem

[AoS, 51 (2023) 1159]

$$\begin{pmatrix} \boxed{X^{(n)}} \\ \Delta^{(n)} \end{pmatrix} \overset{\rho_{\theta_0}^{(n)}}{\rightsquigarrow} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma & \tau \\ \tau^* & J \end{pmatrix} \right)$$

Assume that

1) $\left\{ \rho_\theta^{(n)} \right\}$ is q-LAN and D-extendible at θ_0

2) Seq. of POVMs $M^{(n)} = \{M^{(n)}(B)\}_{B \in \mathcal{B}(\mathbb{R}^s)}$ enjoys

$$\mathrm{Tr}\, \rho_{\theta_0 + h/\sqrt{n}}^{(n)} M^{(n)} \overset{h}{\rightsquigarrow} \exists \mathcal{L}_h$$

Then $\exists M^{(\infty)} = \{M^{(\infty)}(B)\}_{B \in \mathcal{B}(\mathbb{R}^s)}$ on $\mathrm{CCR}(\mathrm{Im}\,\Sigma)$ s.t.

$$\phi_h(M^{(\infty)}(B)) = \mathcal{L}_h(B) \qquad (\forall h)$$

where $\phi_h \sim N((\mathrm{Re}\,\tau)\, h, \Sigma)$

Applications

- asymptotic quantum representation theorem converts a statistical problem for the local parameter h into another one for the limiting q-Gaussian shift model
- asymptotic representation bound beyond iid
- asymptotic regularity and asymptotic minimax theorems that exclude quantum superefficiency

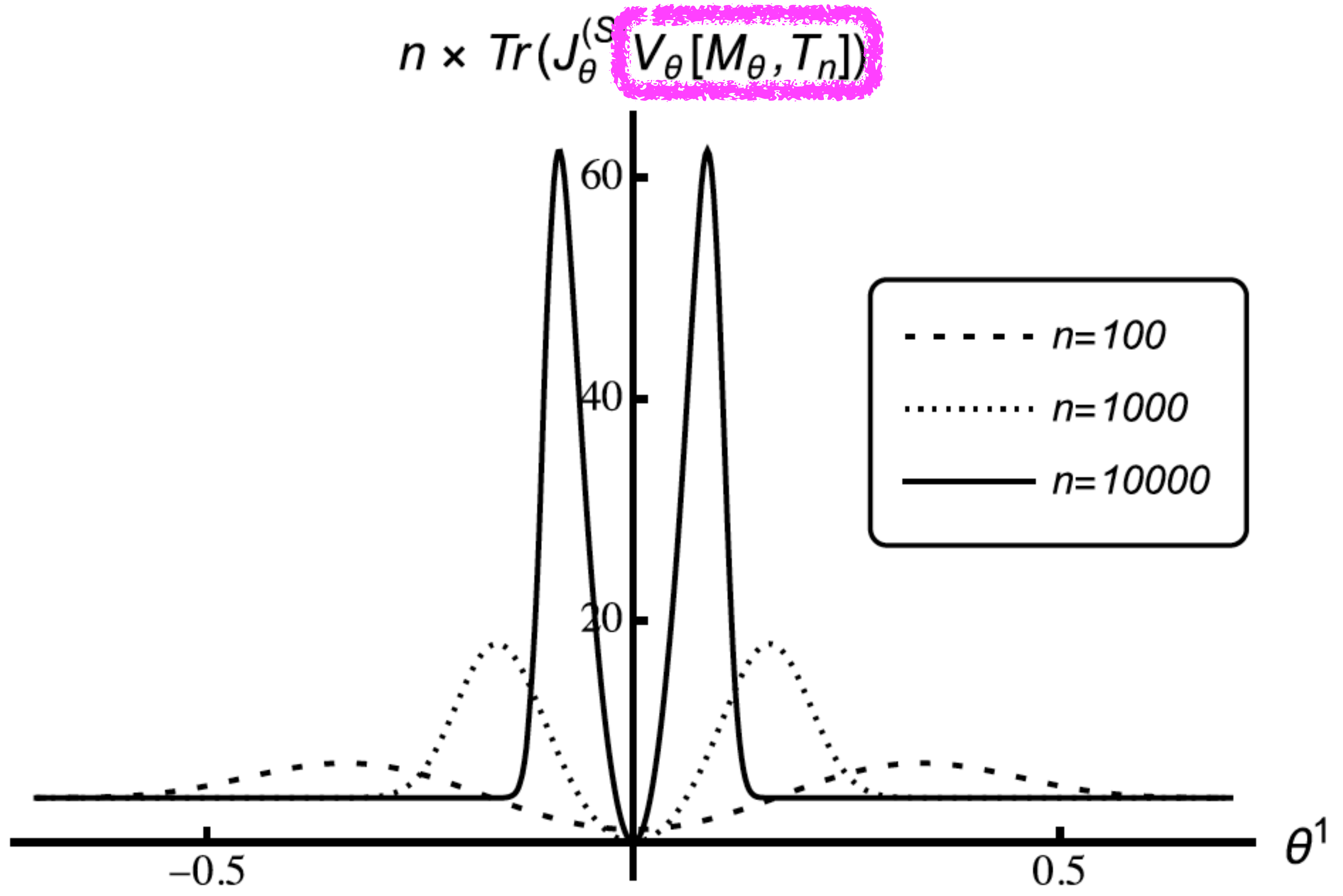


FIG. 1. *Weighted trace of covariance matrix of the quantum Hodges estimator* T_n *with the weight* $J_\theta^{(S)}$ *for the spin coherent state model* $\mathcal{S}$, *based on the means of samples of size* 100 *(dashed)*, 1000 *(dotted), and* 10,000 *(solid) observations. For reference, the corresponding Holevo bound is* $c^{(H)}_{J_\theta^{(S)}} = 4$.

Summary

- Quantum information geometry naturally led us to the quantum Lebesgue decomposition
- With additional notions (such as quantum weak convergence and quantum contiguity), we derived quantum LeCam third lemma and asymptotic quantum representation theorem
- They establish solid foundations of the theory of quantum local asymptotic normality, providing a powerful tool to cope with asymptotics in the quantum domain

"Thank you for your attention"

- Akio Fujiwara