Application of Lattice to Convolutional Codes: Signal Codes and Turbo Signal Codes
（格子の畳込み符号への応用）

電子情報通信学会ソサイエティ大会
2015年9月11日

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Outline

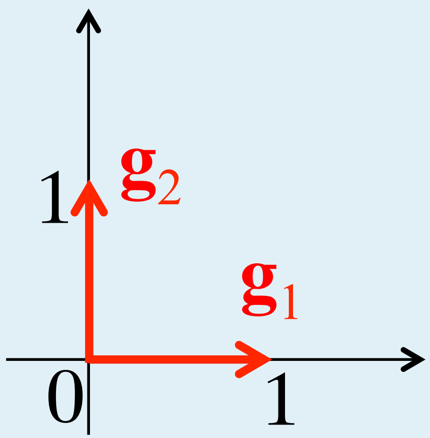
- A very brief review of lattices
- Convolutional lattice codes (Signal codes)
  - Shalvi, Feder & Sommer, "Signal codes: Convolutional lattice codes," IEEE Trans. Inform. Theory, August 2011.
- Recursive convolutional lattice codes and their parallel concatenation (Turbo signal codes)
  - Mitran & Ochiai, "Parallel concatenated convolutional lattice codes with constrained states," IEEE Trans. Commun., March 2015.
- Some performance comparison
- Conclusions

Lattice (1)

- Let $\mathbf{g}_1$ and $\mathbf{g}_2$ denote the n-dimensional (real-valued) vectors that are linearly independent.
  - $\mathbf{g}_1$ and $\mathbf{g}_2$ are called the basis vectors of the n (real) dimensional Euclidean space.

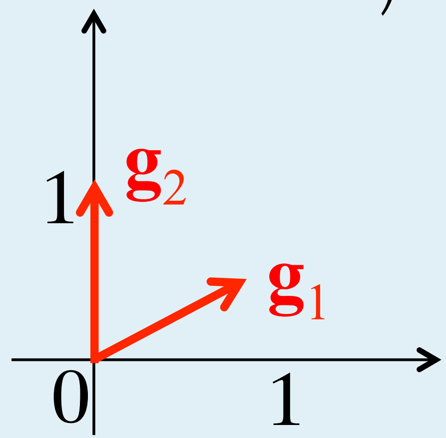
$n = 2$

$\mathbf{g}_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}^T, \quad \mathbf{g}_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$



$\mathbf{g}_1 = \begin{pmatrix} \cos 30^\circ & \sin 30^\circ \end{pmatrix}^T,$

$\mathbf{g}_2 = \begin{pmatrix} \cos 90^\circ & \sin 90^\circ \end{pmatrix}^T$



Lattice (2)

- Two-dimensional lattice Λ is given by a set of real-valued points that are specified by linear combination of the basis vectors with integer coefficient.

$$\Lambda = \left\{ \lambda = \sum_{k=1}^{2} a_k \mathbf{g}_k : a_k \in \mathbb{Z} \right\}$$

$$= \left\{ \lambda = \mathbf{G}\,\mathbf{a} : \mathbf{a} \in \mathbb{Z}^2 \right\} \quad \text{where} \quad \mathbf{G} = \left(\mathbf{g}_1 \,\middle|\, \mathbf{g}_2 \right)$$

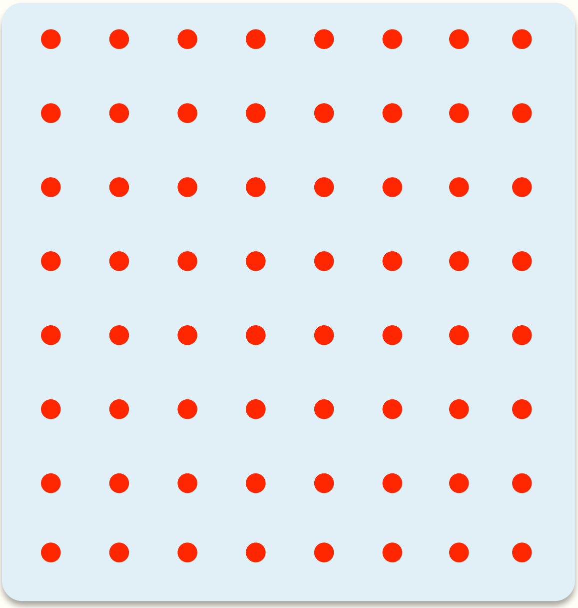
➡ $\mathbf{G}$ is a generator matrix of lattice

Note that $\operatorname{rank}(\mathbf{G}) = 2,\ \det(\mathbf{G}) \neq 0$

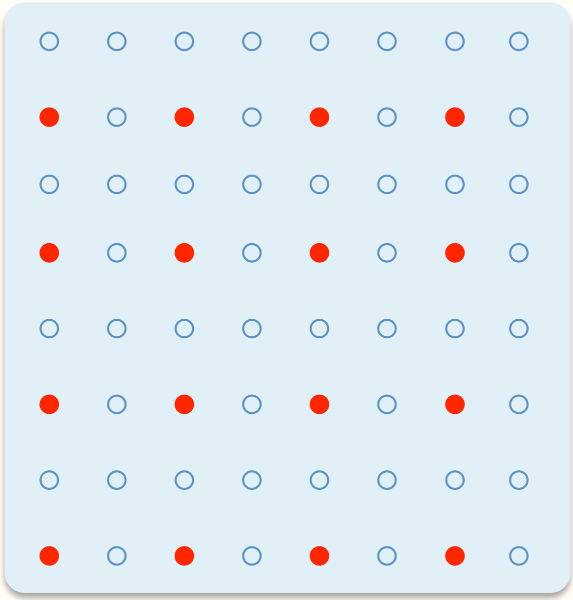
Lattices and Sublattices

- For a given lattice Λ, its subset $\Lambda' \subset \Lambda$ is called sublattice.

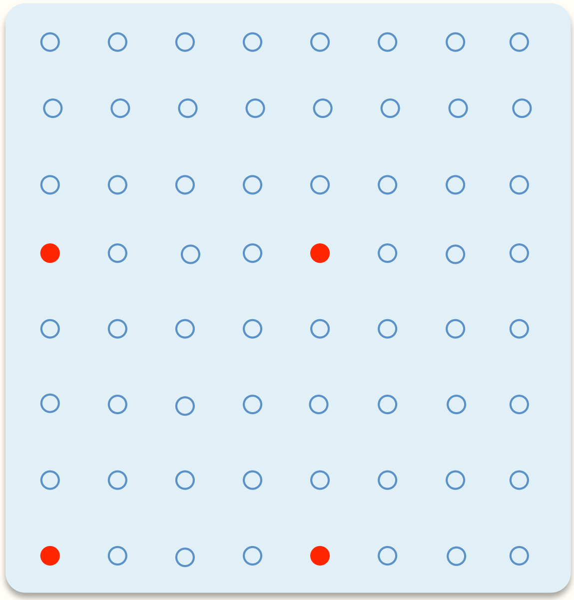
$\Lambda = \mathbf{Z}^2$



$\Lambda' = 2\mathbf{Z}^2$



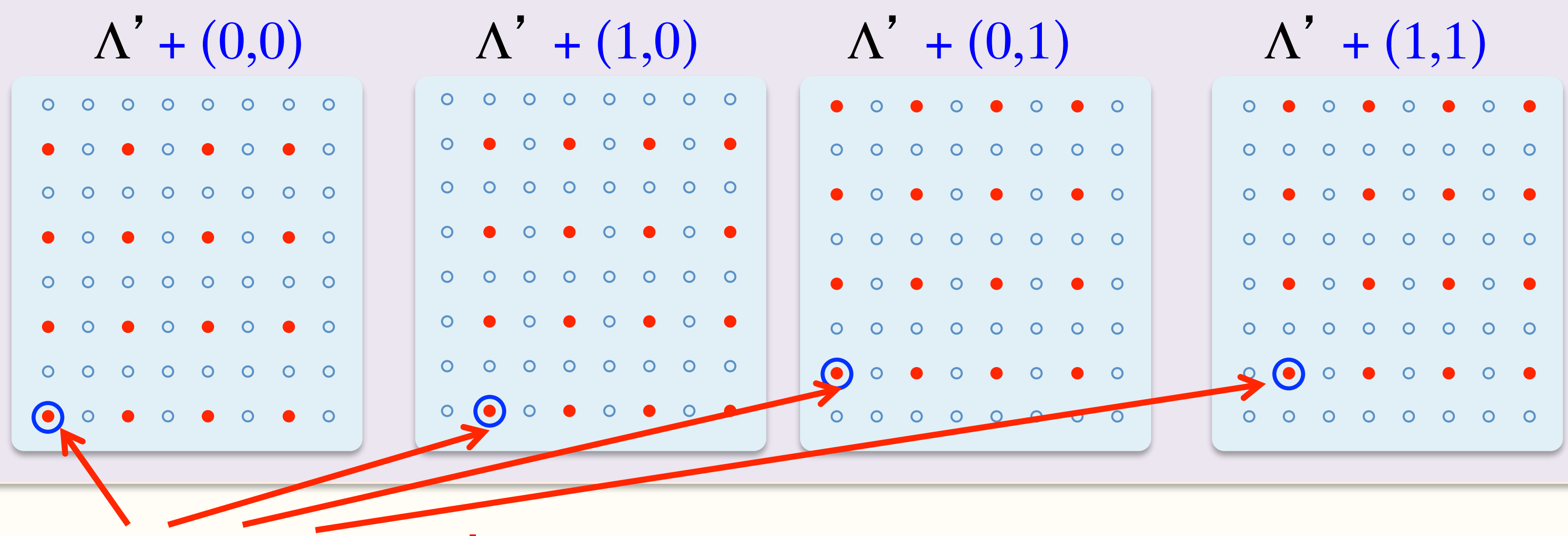
$\Lambda'' = 4\mathbf{Z}^2$



Partition of Lattice

- Λ' induces a partition of Λ, i.e., Λ / Λ', into equivalent classes modulo Λ' (quotient group).
- Each equivalent class is called a coset of Λ'.
- The number of the cosets, $|\Lambda / \Lambda'|$, is called the order of the quotient group.

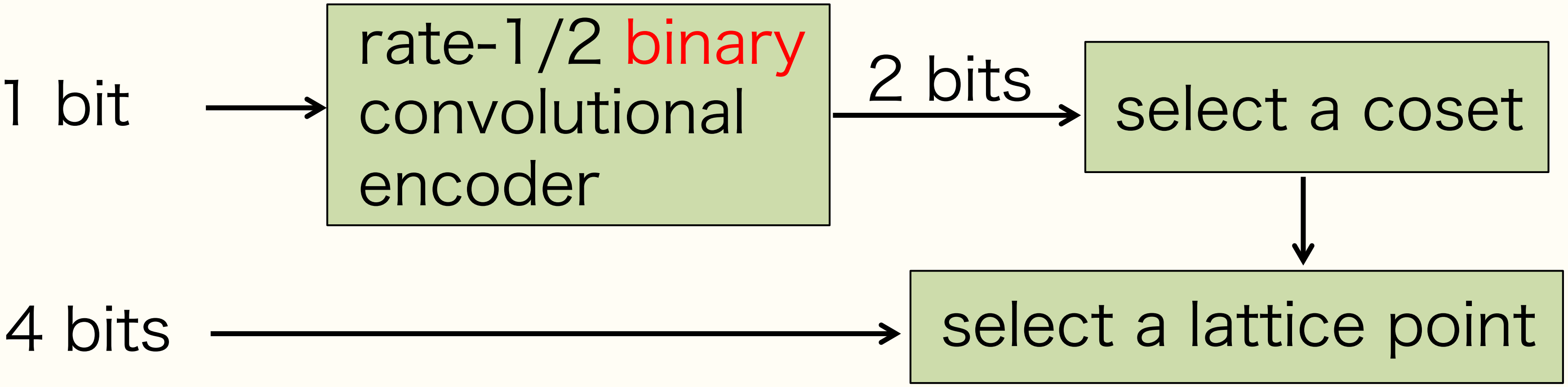
cosets of Λ' for $\mathbf{Z}^2/2\mathbf{Z}^2$



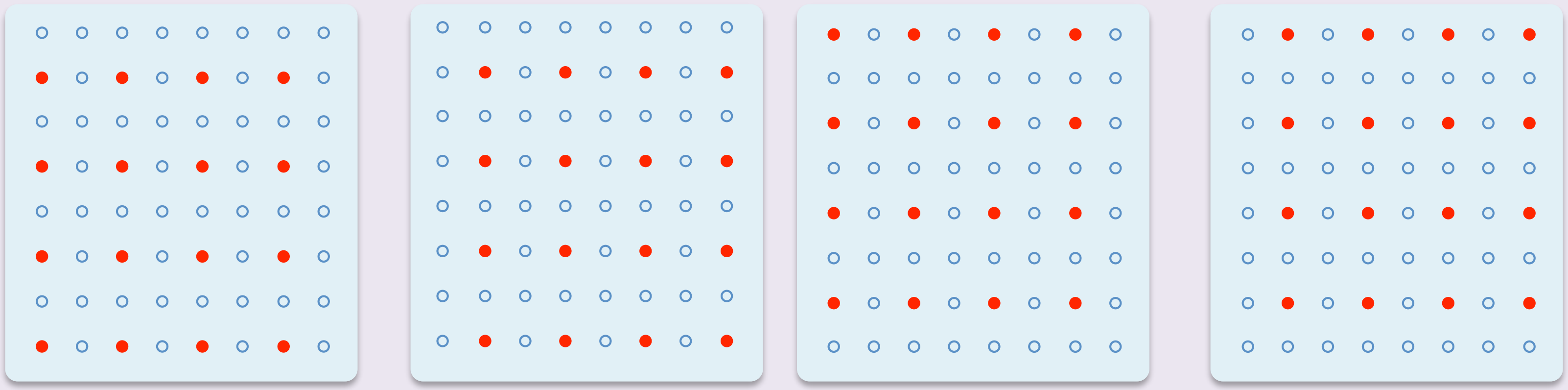
coset representative

A Practical Example of Lattice Codes

- Ungerboeck's 2-D 64-QAM TCM (coset code):



$$|\mathbf{Z}^2/2\mathbf{Z}^2| = 4$$



Coset Decomposition

- If we denote a set of coset representatives by $[\Lambda/\Lambda']$, then each lattice point λ of Λ is expressed with respect to the point λ' of the sublattice Λ' as

$$\lambda \in \Lambda, \lambda' \in \Lambda' \Rightarrow \lambda = \lambda' + \mathbf{c}, \quad \mathbf{c} \in [\Lambda / \Lambda']$$

or alternatively

$$\Lambda = \Lambda' + [\Lambda / \Lambda']$$

$$\mathbb{Z}^2 = 2\mathbb{Z}^2 + \{(0,0),(0,1),(1,0),(1,1)\}$$

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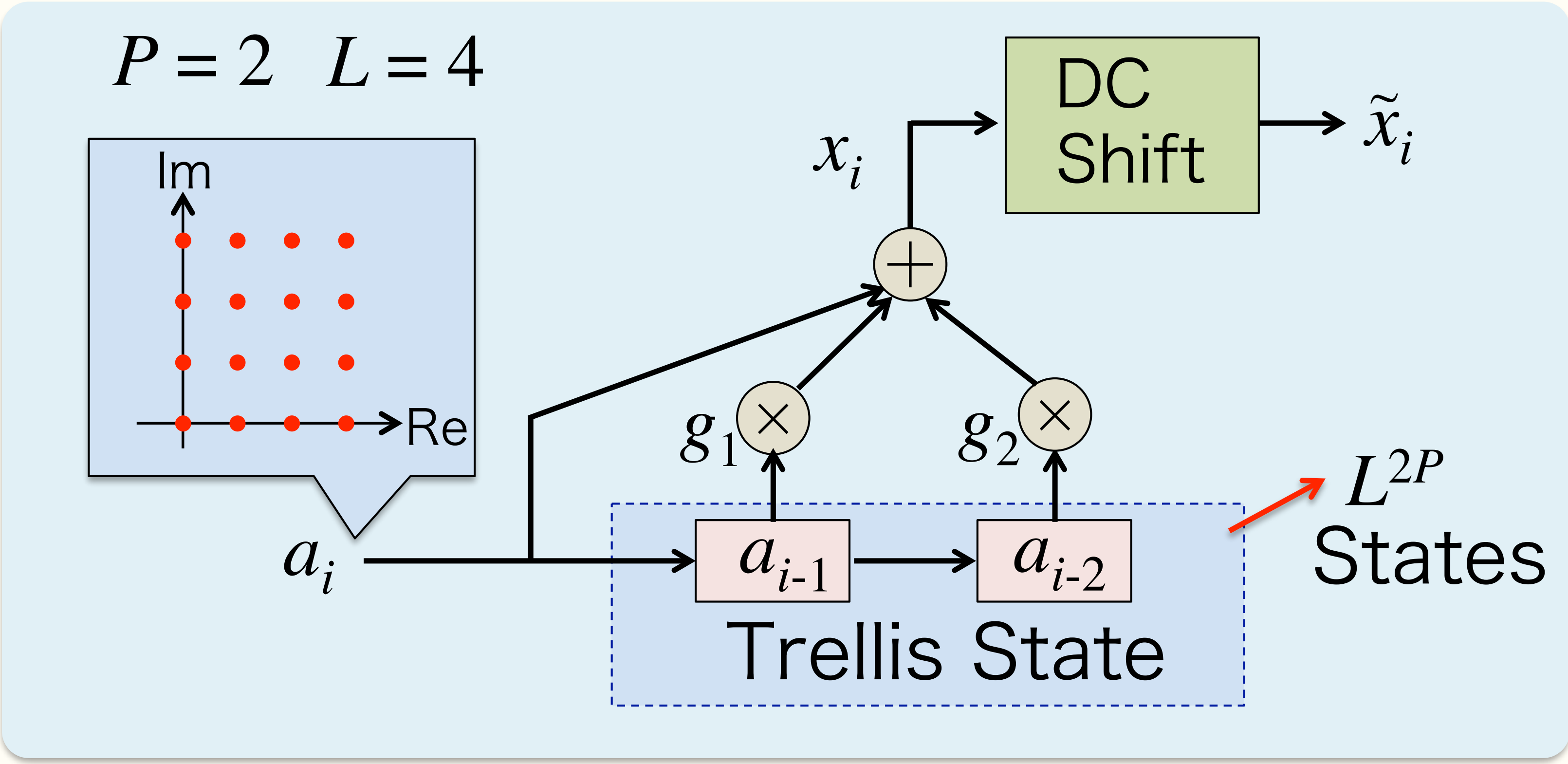
Signal Codes (1)

- Convolutional lattice codes (Shalvi 2011)
- Input: L^2-QAM constellation (uncoded)
- For memory size P, constraint length $K = P+1$



Signal Codes (2)

- Lattice structure

Example: $P=2,\ k=3,\ n=k+P=5$

$$\Lambda=\left\{\lambda=\mathbf{G}\,\mathbf{a}:\ \mathbf{a}\in\left(\mathbb{Z}_L[j]\right)^3\right\}$$

where

$$\mathbf{G}=\underbrace{\begin{pmatrix}1&0&0\\g_1&1&0\\g_2&g_1&1\\0&g_2&g_1\\0&0&g_2\end{pmatrix}}_{k}\qquad g_1,g_2\in\mathbf{C}$$

Signal Codes (3)

Average Power: $P_{\mathrm{av}} = E\left\{|\tilde{x}_i|^2\right\} = 1 + \sum_{k=1}^{2} |g_k|^2$

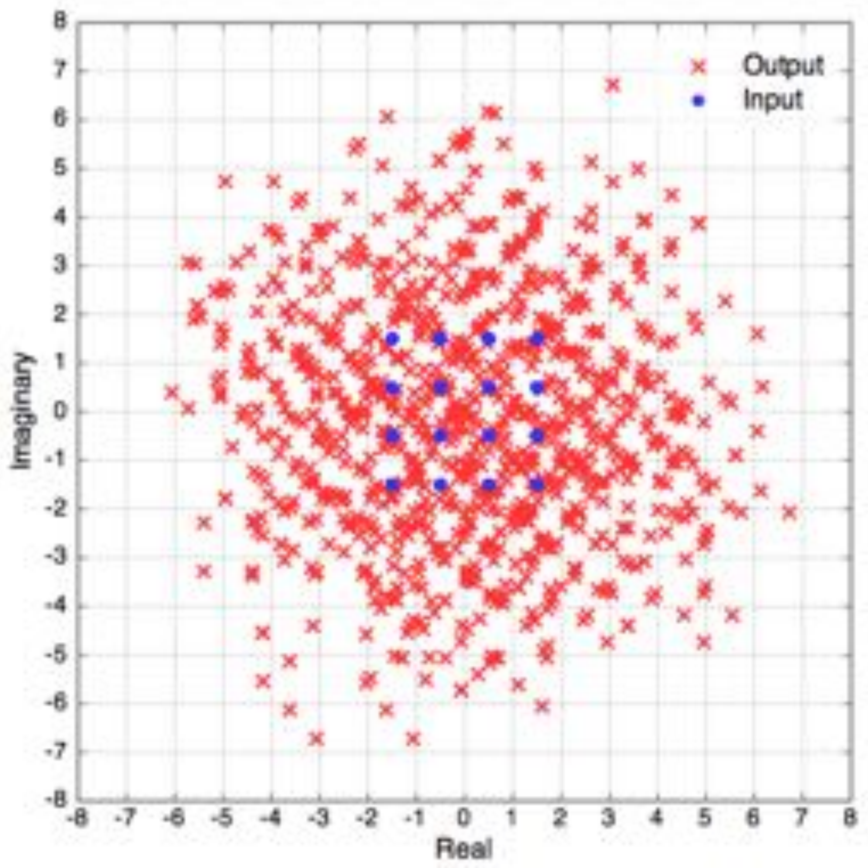
FIR tap

$$G(z) = \left(1 - z_0 z^{-1}\right)^P$$

$$z_0 = -0.90 e^{j0.12\pi},\ P = 2$$

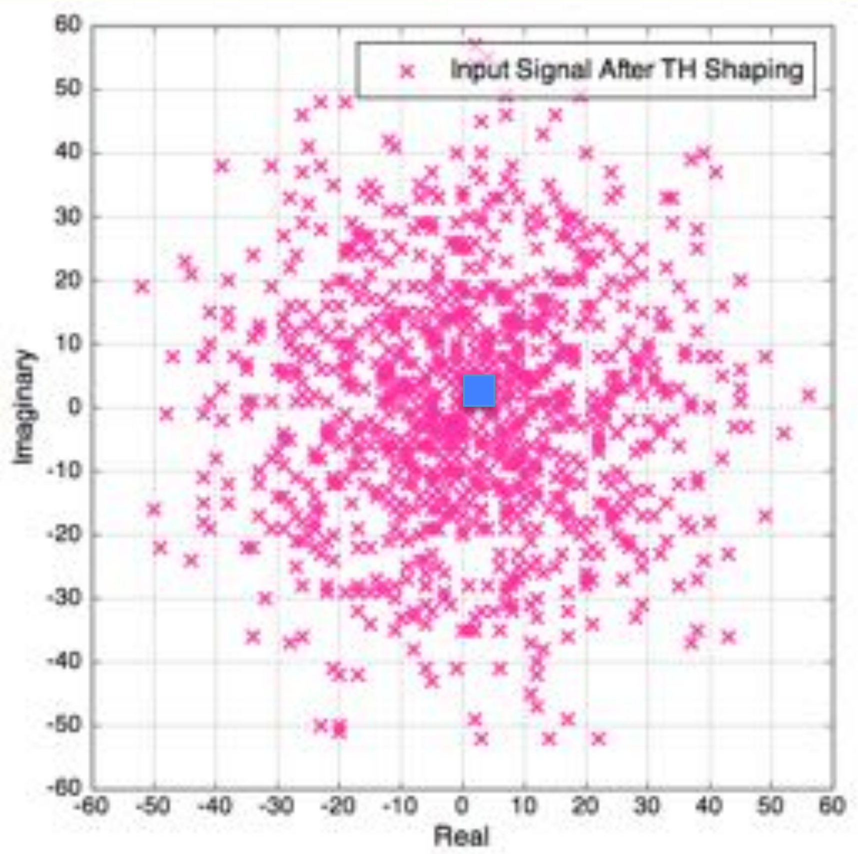
※ Poles close to a unit circle yield high coding gain

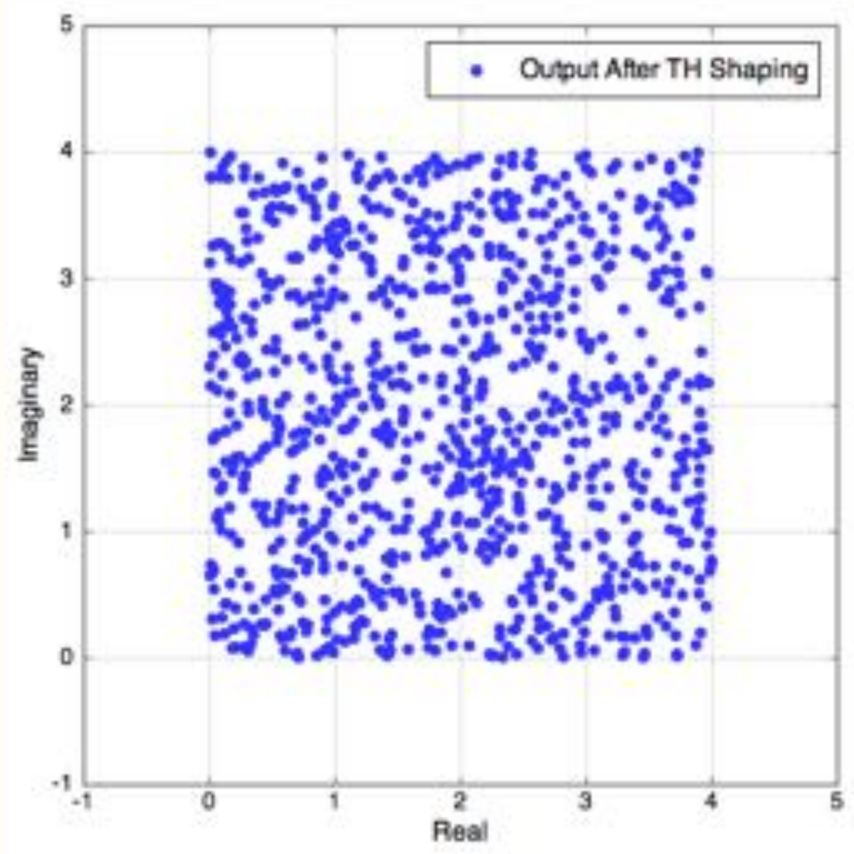
⇨ $P_{\mathrm{av}} = 6.9\mathrm{dB}$



Average power reduction by shaping is essential.

Signal Codes (4)

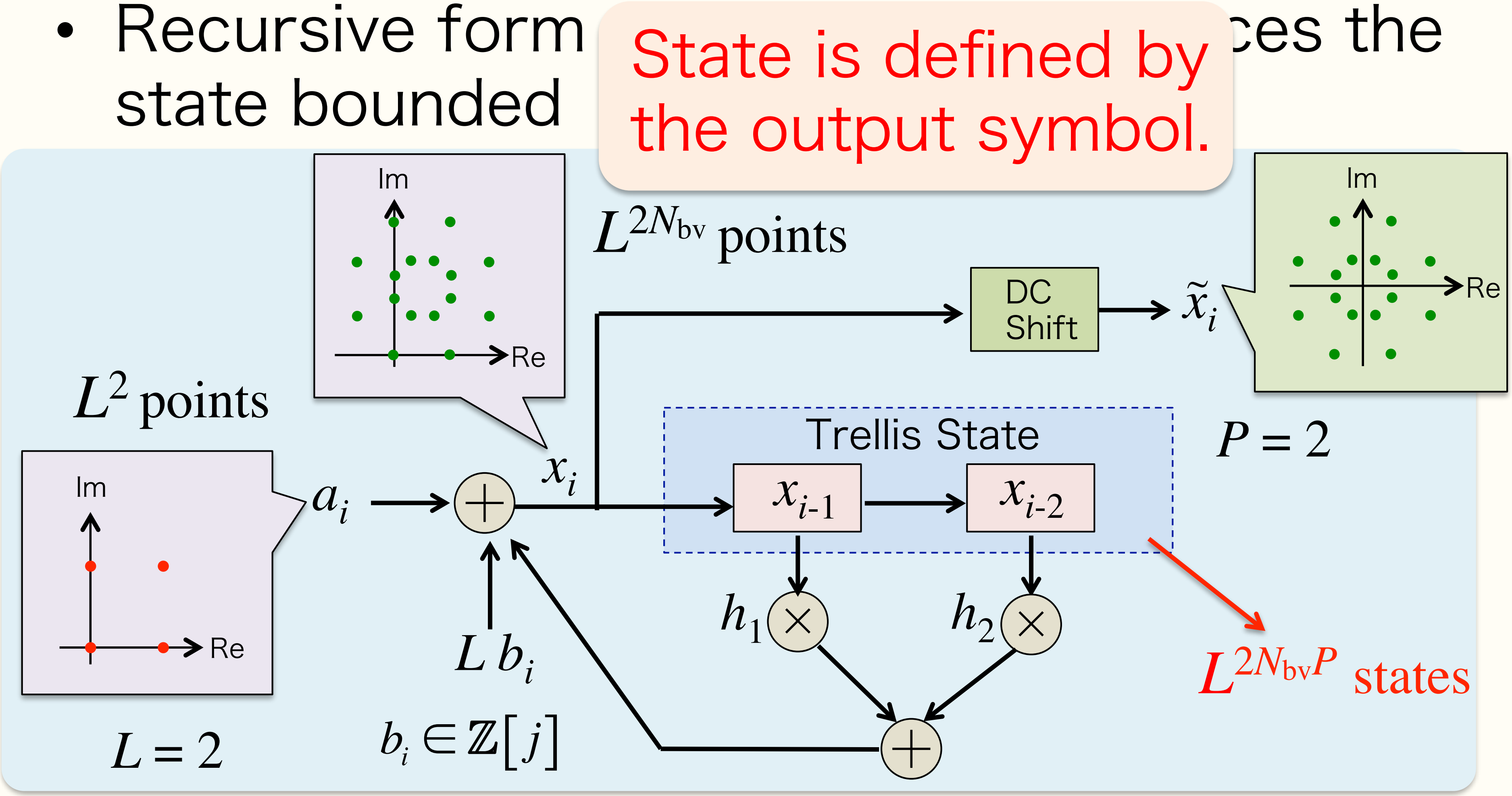




The number of trellis states is unbounded!

Recursive Convolutional Lattice Code (RCLC)

- Convolutional lattice codes with constrained states (Mitran 2015)
- Recursive form ... ces the state bounded



Constellation Design (1)

We consider the following formal power series:

$$\mathbb{Z}[\omega] := \left\{ a_0 + a_1\,\omega + a_2\,\omega^2 + \cdots + a_{n-1}\,\omega^{n-1} : a_k \in \mathbb{Z} \right\}, \quad \omega = e^{j\frac{\pi}{n}}$$

which has the following ring property
(note: $\omega^n = -\omega$)

$$\alpha, \beta \in \mathbb{Z}[\omega] \Rightarrow \alpha + \beta \in \mathbb{Z}[\omega], \alpha\beta \in \mathbb{Z}[\omega]$$

Since $e^{j\pi/2} = j$, for even values of $n = 2N_{\mathrm{bv}}$, we have

$$\begin{aligned}\mathbb{Z}[\omega] &:= \left\{ a_0 + b_0 j + (a_1 + b_1 j)\,\omega + \cdots + \left(a_{N_{\mathrm{bv}}-1} + b_{N_{\mathrm{bv}}-1} j\right)\omega^{N_{\mathrm{bv}}-1} : a_k, b_k \in \mathbb{Z} \right\} \\ &= \left\{ c_0 + c_1\omega + \cdots + c_{N_{\mathrm{bv}}-1}\,\omega^{N_{\mathrm{bv}}-1} : c_k \in \mathbb{Z}[j] \right\}, \quad \omega = e^{j\pi/2N_{\mathrm{bv}}}\end{aligned}$$

Constellation Design (2)

To limit the size of signal points, we further put a constraint that a_k and b_k are integer rings (i.e., coset leaders of the following quotient):

$$C(L, N_{\mathrm{bv}}) := \mathbb{Z}[\omega] / L\mathbb{Z}[\omega]$$

$$= \left\{ a_0 + b_0 j + (a_1 + b_1 j)\,\omega + \cdots + (a_{N_{\mathrm{bv}}-1} + b_{N_{\mathrm{bv}}-1} j)\,\omega^{N_{\mathrm{bv}}-1} : a_k, b_k \in \mathbb{Z}_L \right\}$$

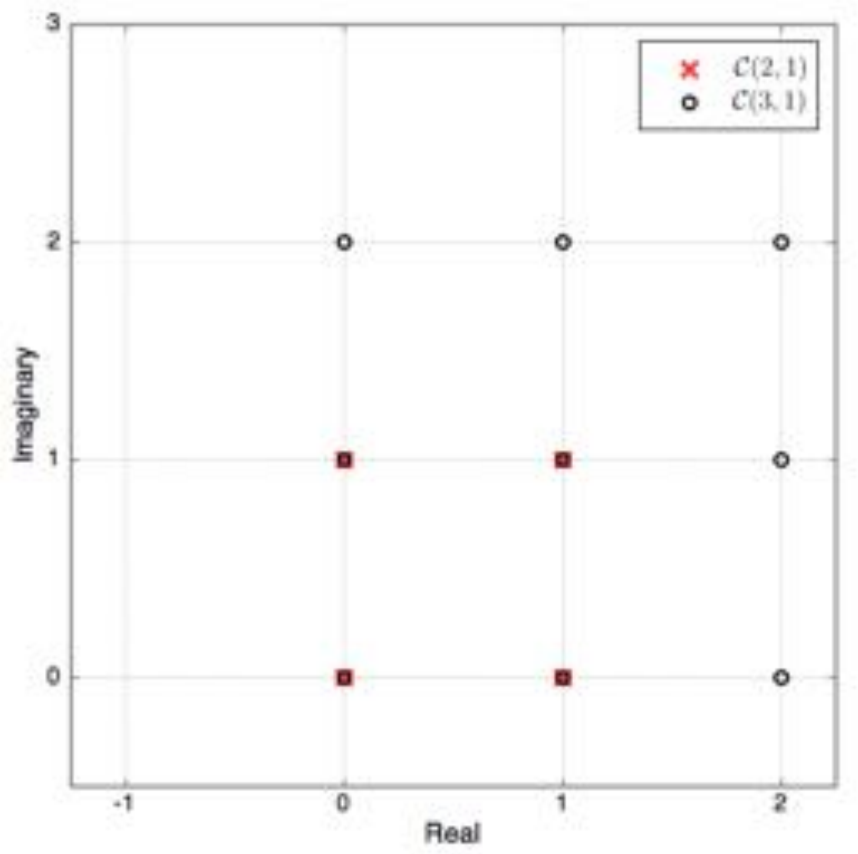
$$= \left\{ c_0 + c_1\,\omega + \cdots + c_{N_{\mathrm{bv}}-1}\,\omega^{N_{\mathrm{bv}}-1} : c_k \in \mathbb{Z}_L[j] \right\}, \quad \omega = e^{j\pi/2N_{\mathrm{bv}}}$$

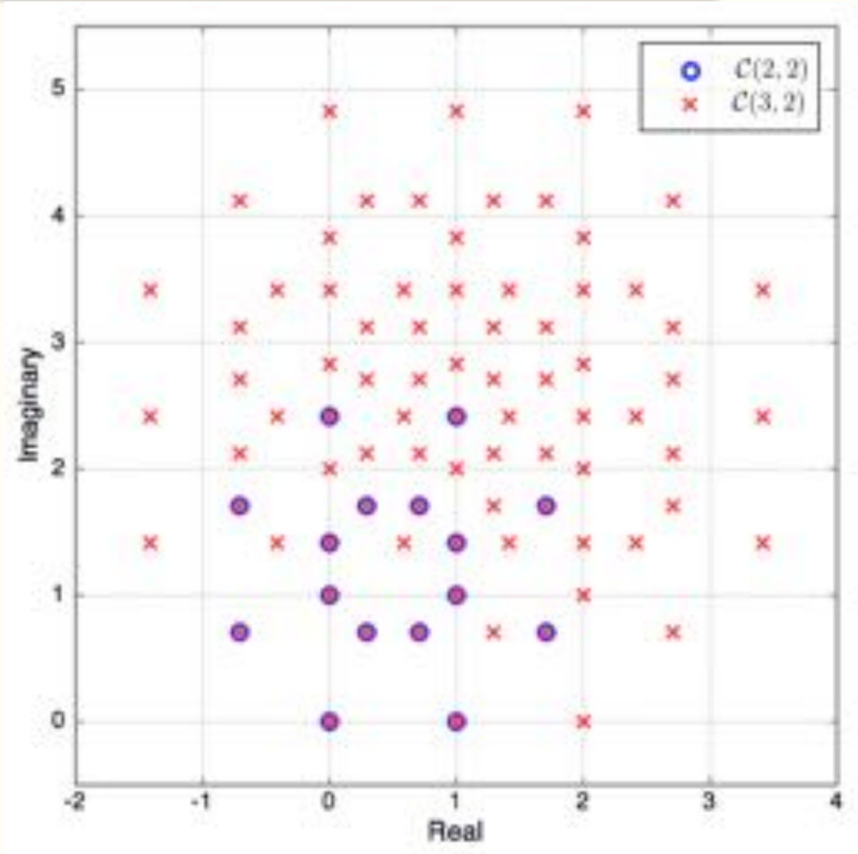
⇨ $$C(L, N_{\mathrm{bv}}) \subset \mathbb{Z}\left[e^{j\pi/2N_{\mathrm{bv}}}\right]$$

Constellation Design (3)

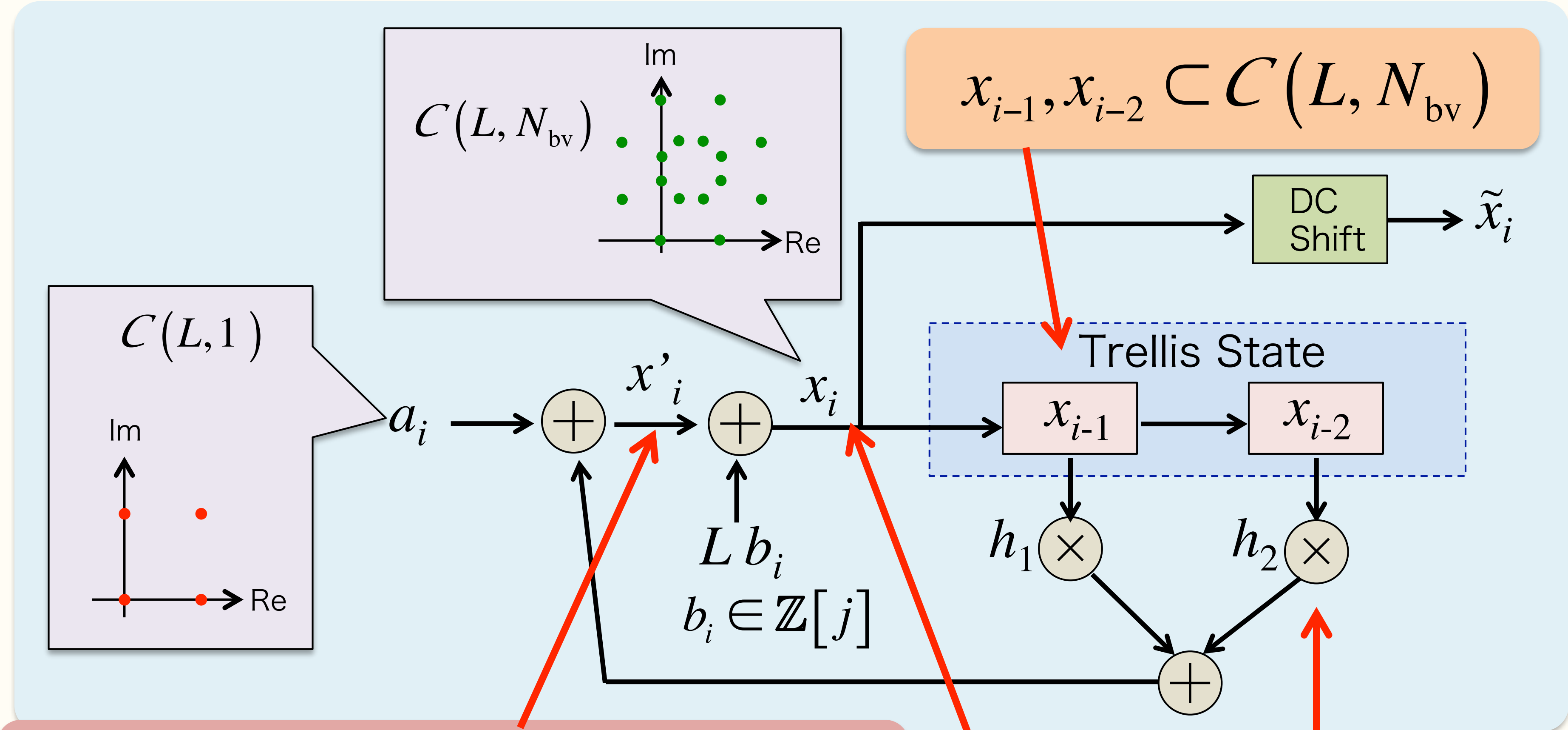
In general, for $N_{\text{bv}} > 1$, we have

$$\mathcal{C}(L,1) \subset \mathcal{C}(L, N_{\text{bv}}) \subset \mathcal{C}(L+1, N_{\text{bv}}) \subset \mathbb{Z}\left[e^{j\pi/2N_{\text{bv}}}\right]$$





State-Constrained Signal Codes



$$x_{i-1}, x_{i-2} \subset C(L, N_{\mathrm{bv}})$$

$$x'_i = a_i + h_1 x_{i-1} + h_2 x_{i-2} \subset \mathbb{Z}\left[e^{j\pi/2N_{\mathrm{bv}}}\right]$$

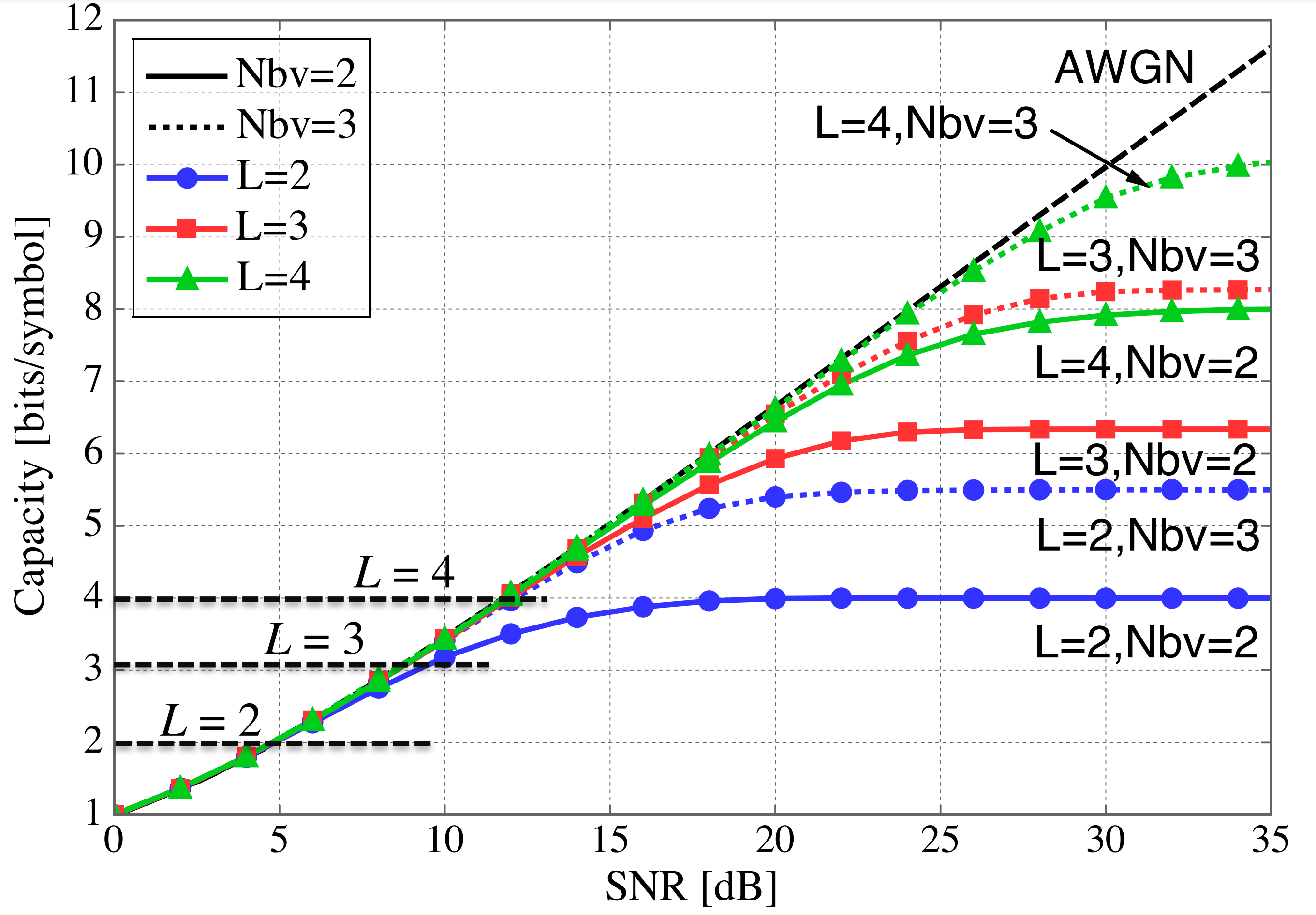
$$h_1, h_2 \subset C(L, N_{\mathrm{bv}})$$

$$x_i = x'_i + L\, b_i \subset C(L, N_{\mathrm{bv}})$$

Constrained Capacity

Information rate (bits): $\log_2 |C(L,1)| = \log_2 L^2 = 2\log_2 L$

Constellation size: $\log_2 |C(L,N_{\text{bv}})| = \log_2 L^{2N_{\text{bv}}} = 2N_{\text{bv}} \log_2 L$



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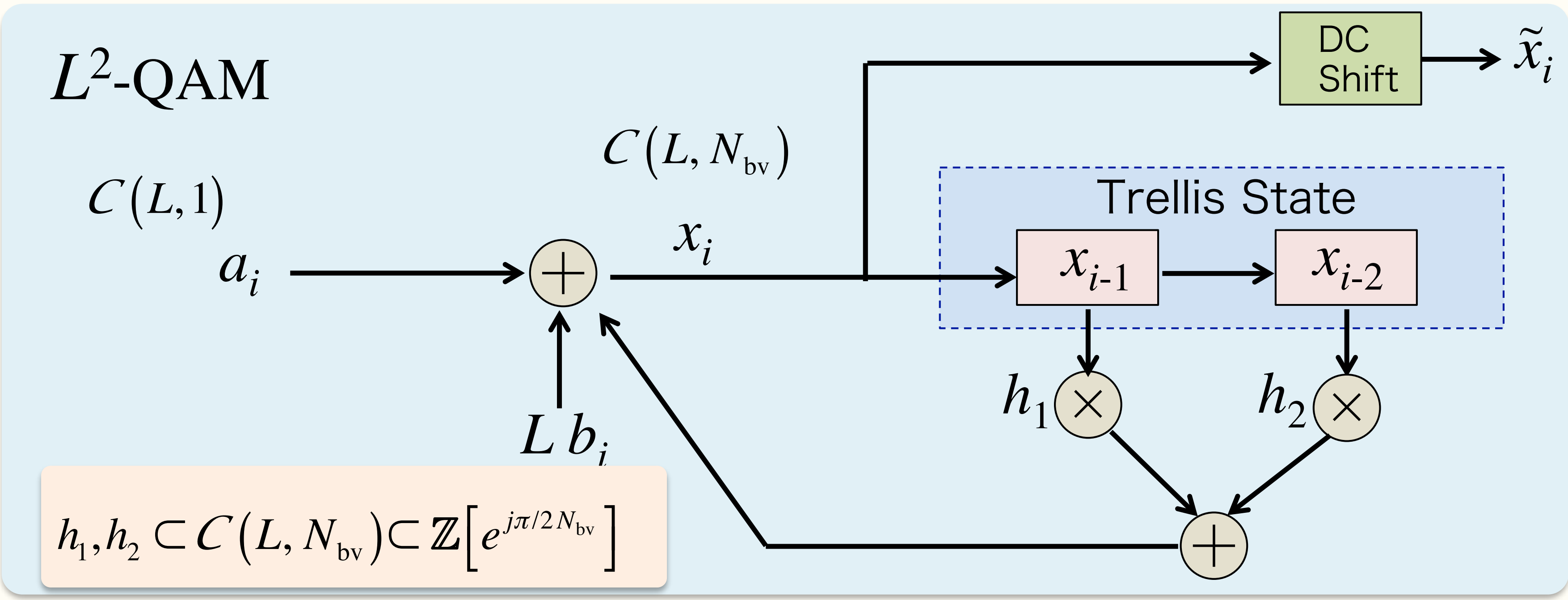
Some Remarks

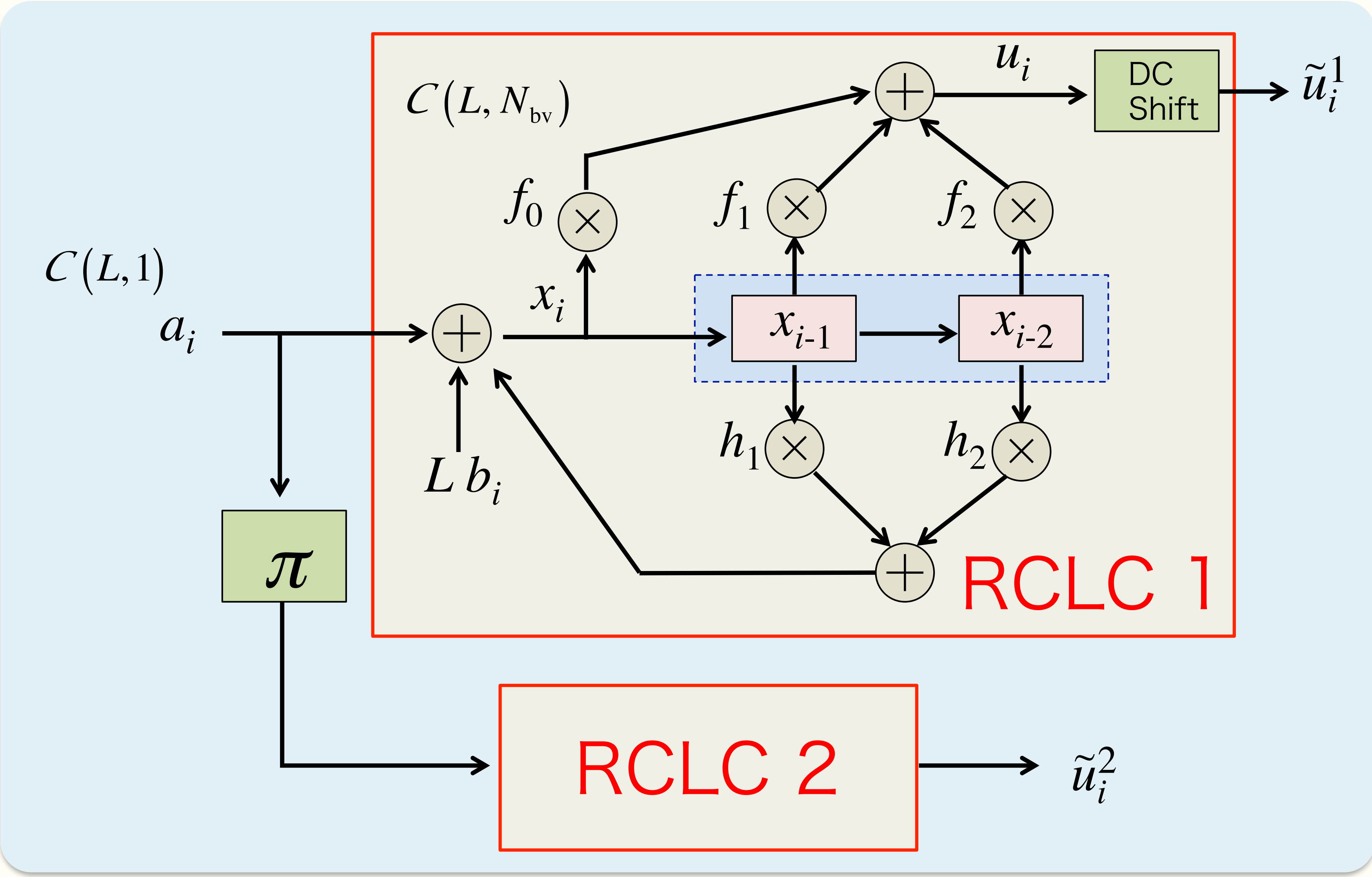
- The information rate is $2 \log_2 L$ bits per 2D.
- Code (tap) selection: Unlike the conventional lattice codes, it is difficult to analyze minimum Euclidean distance due to the lack of regularity, resorting to brute-force search



Turbo Signal Codes

- By constraining the state size ($L^{2N_{\mathrm{bv}}P}$ states), trellis based decoding such as Viterbi algorithm can be now employed.
- However, the performance may not be significant because it is a variant of recursive convolutional codes.
- Due to the limited number of states, the BCJR decoding can be used.
- Since the code is recursive, can we take an approach similar to binary turbo codes and perform MAP decoding?

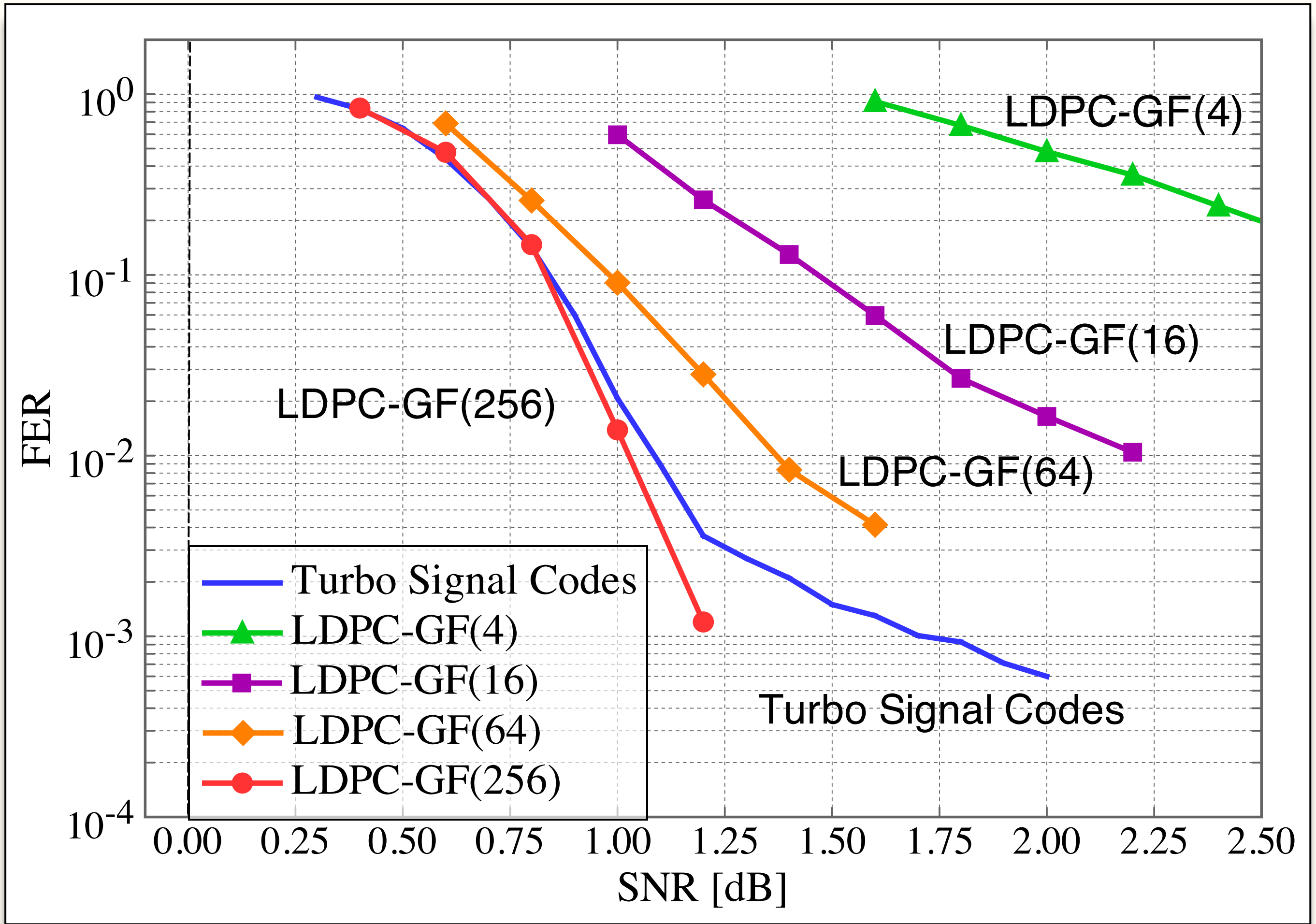
Turbo Signal Codes



Comparison with Non-binary LDPC

Lattice: $P = 1, L = 2, N_{bv} = 2$
LDPC: QPSK, rate = 1/2

(frame length 4096 bits)



松峯, 落合, RCS研究会 (2015年8月)

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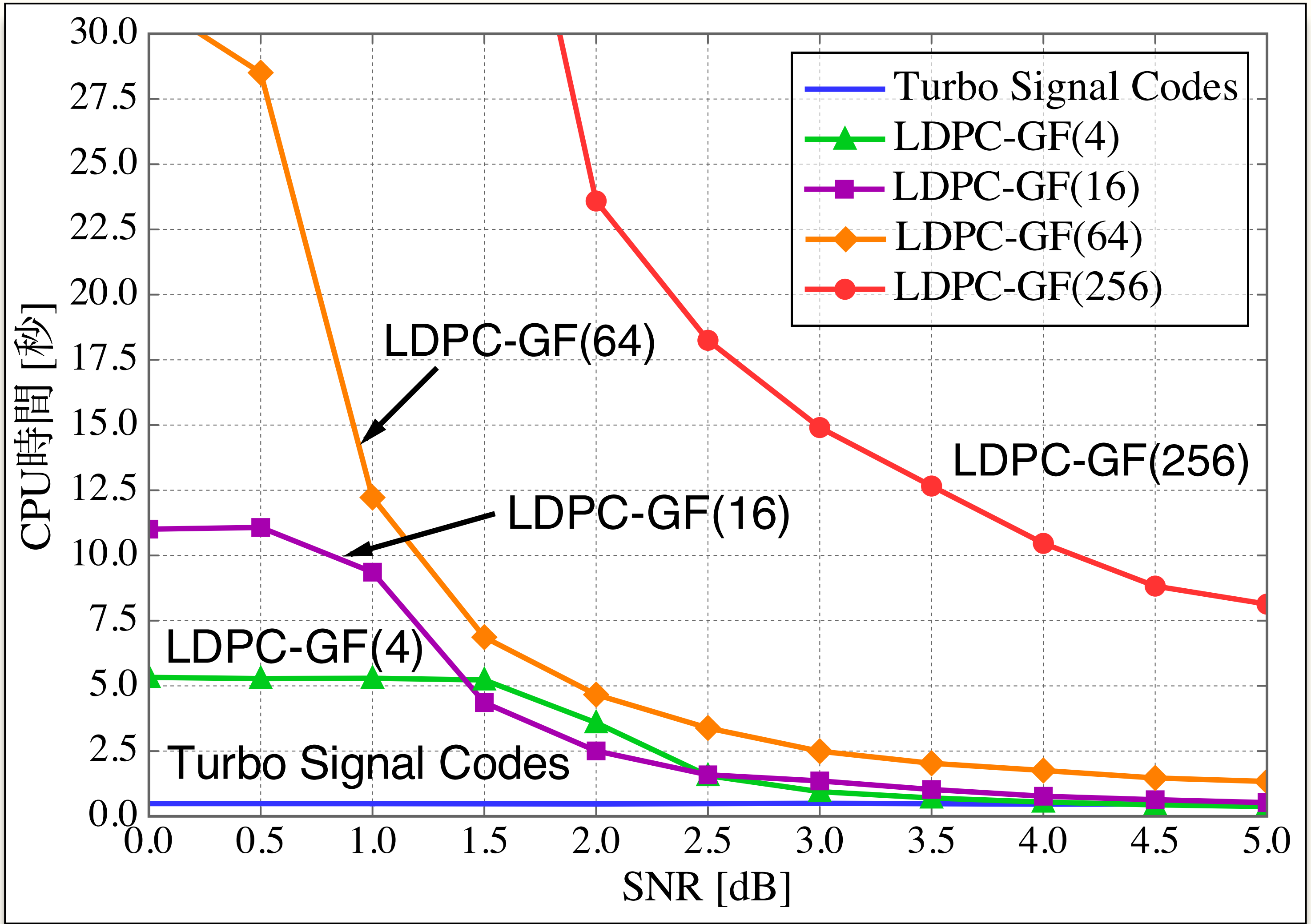
Comparison of Decoding Complexity



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Conclusions

- We have considered application of lattice in an unconventional scenario:
  - Signal Codes
  - Turbo Signal Codes
- Due to the lack of structure, optimal design of coding (filter taps) is challenging from a theoretical viewpoint
- There are many issues unknown:
  - Performance analysis and code design
  - Extension to even higher constellation size
  - Complexity vs. performance trade-off
  - Puncturing for higher rate